MATHEMATICS

THE PUPIL-TEACHER'S COURSE

OF

MATHEMATICS.

PART II.

LONDON : PRINTED BY
SPOTTISWOODE AND CO., NEW-STREET SQUARE
AND PARLIAMENT STREET

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MATHEMATICS.

PART II.

ALGEBRA TO THE END OF QUADRATIC EQUATIONS.

BY

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LONDON:
NATIONAL SOCIETY'S DEPOSITORY,
WESTMINSTER.
1879.

PREFACE.

In most works on Algebra the student is too soon made to face the difficulties of *Symbolical* Algebra; it is far better to defer this until *Arithmetical* Algebra has been mastered up to the end of quadratic equations. Accordingly in this treatise the symbols will denote the numbers and operations of arithmetic only, which will greatly simplify our proofs of the rules. The theory of indices and surds (in which we interpret the symbols and operations so as to make the rules of arithmetical algebra hold universally), and impossible quantities, are thus left for the student's consideration after he has read through the present book, when he will be able to handle algebraical symbols and formulæ with precision and rapidity. This done, his future progress will be easy. Thus it is hoped that this little book will be useful to all who are beginning to read algebra.

It is, however, especially intended for the use of pupil-teachers, and the course they are required to study is exactly contained in this work. Pupil-

teachers of the fourth* year are examined in the first four rules, greatest common measure, least common multiple, fractions, square root, and simple equations involving one unknown quantity, with easy problems. The course for the fifth † year is the same, with cube root, simple equations with more than one unknown quantity, quadratics involving one unknown quantity, and easy problems. The course for the fourth year is comprised in the first six chapters: the sections on cube root may be omitted in that year.

The fifth year pupil-teachers should study, in addition, cube root and the last two chapters. Fifty-two examination papers, with some suggestions as to the manner of working them, are given at the end of the book.

* Or, *third*, according to date of admission. See first schedule, New Code.

† Or, *fourth*.

PUPIL-TEACHER'S

COURSE OF MATHEMATICS.

PART II.

ALGEBRA.

CHAPTER I.

First Ideas, Definitions, and Explanation of Signs.

1. In Algebra we reason about quantities (or numbers) by means of letters and signs, which represent the numbers themselves, and the manner in which they are related to others.

In symbolical algebra we determine what meanings must be given to the signs and symbols when the rules of arithmetical algebra are true, not for numbers only, but universally.

In the present course we shall be concerned little or not at all with this extended view of the subject; our letters and symbols will denote numbers and operations dealt with in arithmetic.

Thus algebra is the science of generalisation, as regards number (or quantity). In arithmetic, for example, we learn that the sum of 7 and 5 multiplied by their difference is equal to the difference of their squares; but algebra teaches us *generally*, that the sum of any two

numbers whatever, multiplied by their difference, is equal to the difference of their squares.

2. The reader has already learnt the meaning of the signs $\therefore$, $\because$, $=$. The sign $+$ (read plus) signifies that the quantity before which it stands must be added; and the sign $-$ (minus) that the quantity before which it stands must be subtracted. Thus $a+b$ means that the quantity denoted by b is to be added to the quantity denoted by a; $a-b$ that b is to be subtracted from a.

3. All quantities before which $+$ stands are called *positive*; all before which $-$ stands are called *negative* quantities. When neither $+$ nor $-$ is placed before a quantity, it is understood to be positive.

4. The sign $\times$ signifies that the quantities between which it stands are to be multiplied together. This sign is often omitted, or a point put for it; thus $a\times b$, $a \,.\, b$, $a\,b$, mean one and the same thing, viz. a multiplied by b.

5. The number (or letter), whether positive or negative, prefixed to any algebraical quantity, is called its coefficient; thus 6 is the coefficient of $6a$, a of ax, and so on. The former coefficient is termed *numerical*, the latter *literal* (consisting of a letter or letters).

6. Any collection of letters and signs is called an *expression* or *formula*, and the parts of an expression connected by $+$ or $-$ are its terms; when there are two terms, it is called *binomial*; when there are three, *trinomial*; or generally, when there are more than two terms, it is said to be *multinomial.*

7. Algebraical terms are said to be *like* or *similar* when they differ only in their coefficients; otherwise they are said to be *unlike.* Thus $4ab$, $9ab$ are like; $4ab$, $9bc$ are unlike.

8. Any quantity, which as a multiplier serves to make up a product, is called a *factor* of that product. Thus

in $4ab$; 4, a, b are each of them factors; so are $4a$, $4b$, ab, the former being *simple*, the latter *compound* factors.

Examples (1).

(1) What kind of an expression is $4ac-bd$? Point out the terms, the numerical coefficient of ac, the literal coefficient of d, and the factors of each term.

(2) It was said (in section 4) that the sign of multiplication is often omitted. Why however can we not omit it between numbers, *e.g.*, what error might result in writing 4×5, 45?

(3) What are the simple factors of $9abc$?

(4) If $a=5$, $b=4$, $c=7$, $d=8$, $e=0$, find the value of $4a+3b+5c+8d+7e$.

(5) Of $5c-3d-2b-14e+6a$.

(6) Of $7e-10a+14b-25c+d+abc$.

(7) Of $3ab+4cd+6de$.

(8) Of $7ab-14ae-5cd-10be$.

(9) Of $8abc+7bcd-6cde$.

(10) Of $4a-6ab+5bcd-7abcd+abcde$.

9. When a quantity is multiplied any number of times by itself, the product is called a *power* of the quantity, and is expressed by writing a little to the right above the quantity a small figure (called the *index* or *exponent* of the power), which shows the number of times it is repeated.

Thus a is called the first power of a (and means a^1, though it is not necessary to write the index in this case); a^2 is the second power (or square) of a, and means $a\times a$; a^3 is the third power (or cube), and means $a\times a\times a$; and so on.

More generally, if n be any number, a^n is the n^{th} power of a, and means a multiplied by itself n times.

The figures 1, 2, . . . n are the *indices* (or *exponents*) of the first, second . . . n^{th} powers of a.

10. The sign of division is $\div$, and it signifies that the former of the two quantities between which it is placed is to be divided by the other. Thus $a \div b$ means a divided by b or $\frac{a}{b}$.

11. The reader has doubtless learnt the meaning of the square root of a number in arithmetic; and that the cube root is the number whose cube is equal to the given number; similarly for the fourth root, and so on. Generally the n^{th} root of a quantity a is that quantity which, when multiplied n times by itself, is equal to a. The sign used to signify *root* is $\sqrt{\ }$; thus $\sqrt[2]{a^2}=a$, $\sqrt[3]{343}=7$, $\sqrt[4]{625}=5$, and so on; clearly too $\sqrt[n]{a^n}=a$. In the case of the square root, the index showing what root is to be taken is usually omitted, so that $\sqrt{a}$ is understood to mean the *square* root of a.

12. When several quantities are taken collectively they are often enclosed by brackets, such as ( ), { }, [ ], &c. Thus $a-(b-c)$ means that the quantity denoted by $b-c$ is to be taken from a; $(a+b+c)(d+e)$ is $a+b+c$ multiplied by $d+e$; $(a-b)^3$ is the cube of $a-b$; $a-\{b-(c-d)\}$ means that $c-d$ is to be taken from b, and the result is to be taken from a. Sometimes a line (called a vinculum) is used instead of a bracket; thus $a-(b-c)$ may be written $a-\overline{b-c}$.

13. The sign $>$ means greater than, and $<$ means less than; thus $a>b$ is read, *a is greater than b.*

14. A quantity is a *multiple* of another quantity when it contains it an *exact* number of times; thus $15a$ is a multiple of $3a$. It is a *measure* of another quantity when it is contained in that quantity exactly; thus $3a$ is a measure of $15a$.

15. Two quantities are *prime* to one another when they have no common measure but unity; thus 17 is prime to 23.

Examples (2).

Find the value of each of the three following expressions when $a=1$, $b=2$, $c=3$, $d=4$, $e=0$.

(1) $a^2+\frac{2b}{c}-\frac{3d}{4c}-e$. (2) $\frac{3ab+4de}{9c}-\frac{d^2-c^2}{2b+e^2}$.

(3) $\frac{a^5-3ab^2c^2+5c^2d^3-e^5}{c^4-b^4}+\frac{6abcde+ad^4}{d^4-c^4}$.

(4) Find the value of $\sqrt{169}+\sqrt[3]{8}+\sqrt[4]{256}$.

(5) Find the value of $6\sqrt[3]{729}+9\sqrt[5]{1}+7\sqrt[4]{2401}$.

(6) If $a=1$, $b=9$, $c=49$, find the value of $6\sqrt{a}-3\sqrt{16b}+12\sqrt{bc}$.

(7) If $a=3$, $b=4$, $c=5$, find the value of $3\sqrt{3ab}-6\sqrt{c^4}+9\sqrt{15abc}$.

(8) If $a=5$, $b=7$, $c=0$, find the value of $3(b-a)^2-4\sqrt{a+b+69+c}+6\{b^2-(c+a)^2\}$.

(9) Also of $6\sqrt{6(b^2-a^2)}-7\sqrt[3]{12(a+b)^2}$.

(10) Find the value of $\frac{a}{b}-\sqrt{\frac{1+a}{1-b}}$ when $a=\frac{1}{4}$, $b=\frac{1}{5}$.

CHAPTER II.

Addition, Subtraction, Brackets, and Negative Quantities.

16. (1) The addition of *unlike* algebraical quantities is performed by connecting them together with their proper signs.

Ex. Add $3a-2b$ to $5d-4e$.

$$\begin{array}{r} 3a-2b \\ 5d-4e \\ \hline 3a+5d-2b-4e. \end{array}$$

The reason is clear; $5d$ added to $3a$ is $3a+5d$, and $2b$ to be subtracted *together with* $4e$ to be subtracted, is $-2b-4e$. It is immaterial in what order we set down the quantities, provided that each has the proper sign. Thus if to a farm be added 7 horses and 4 cows; and then 6 sheep be taken from it, the result is the same as if the cows were first added, then the horses, and then the sheep taken away, or as if the sheep were taken first and the cows and horses were then added, and so on :— thus $7h+4c-6s$, $4c+7h-6s$, $-6s+4c+7h$, are identical in value.

(2) If any terms are *like*, the addition is shortened by uniting them. First when the like terms have the same sign, add the coefficients, prefix the proper sign, and annex the common letter or letters.

Ex. Add $3a-2b+c$ to $4a-6b-2d$.

$$\begin{array}{l} 3a-2b+c \\ \underline{4a-6b-2d} \\ 7a-8b+c-2d \end{array}$$ is the sum.

For $4a$ added to $3a$ is $7a$ to be added; $6b$ to be subtracted with $2b$ to be subtracted is $8b$ to be subtracted; lastly, as in case (1), c is to be added, and $2d$ to be subtracted.

Secondly, when any like terms have *different* signs, to find their sum, take the difference of the coefficients with the sign of the greater, and annex the common letters as before.

Ex. Add $4a+2b-3c$ to $2a-5b+d$.

$$\begin{array}{l} 4a+2b-3c \\ \underline{2a-5b+d} \\ 6a-3b-3c+d \end{array}$$ is the sum.

For as before $4a+2a$ is $6a$; we then have $2b$ to add and $5b$ to subtract, and therefore on the whole $3b$ to subtract; the unlike terms follow as before.

When there are several expressions to be added, the process is just the same: as the *order* of the terms in each is immaterial, we place all the *like* terms one under the other, not that this is essential, but to avoid risk of error.

Ex. Add $5a-4b+3c$, $6b-7a-9c$, $5c-2b+3a$.

$$\begin{array}{r} 5a-4b+3c \\ -7a+6b-9c \\ \underline{3a-2b+5c} \\ a-c \end{array}$$

Examples (3).

Add together:

(1) $3a-2b$, $4b-9a$, $b+6a$.

(2) $5a-3b-2c$, $6a-7c-14b$, $17b+9c-10a$.

(3) $4a^2-6a+10$, $-3a^2+5a-9$, a^2+a-1, $6-2a^2$.

(4) $4ab-5ac+3b^2$, $12ac-6ab+3a^2$, $3c^2-7ac+2ab$, $-2a^2-2b^2-2c^2$.

(5) $4ab^2-\frac{3}{5}a^2b$, $\frac{3}{7}ab^2-6a^2b$.

(6) $9x^3-\frac{3}{4}x^2y+\frac{16}{17}y^3$, $\frac{11}{4}x^2y-\frac{7}{9}x^3-\frac{3}{17}y^3$, $\frac{11}{9}x^3+\frac{1}{2}x^2y-\frac{21}{17}y^3$.

SUBTRACTION.

17. The rule for subtracting algebraical quantities is :—*change their signs and proceed as in addition.*

For if b be taken from a, the result is $a-b$, but if $b-c$, be taken from a, the result will clearly be *greater* than the former by c, since the quantity subtracted is *less* by c; hence $a-(b-c)=a-b+c$, which proves the rule.

Ex. From $3a^2-2ab+5b^2$ take $3b^2+2a^2-3ab$.

$$\begin{array}{r} 3a^2-2ab+5b^2 \\ 2a^2-3ab+3b^2 \\ \hline \text{answer } a^2+ab+2b^2 \end{array}$$

Examples (4).

(1) From $3a-2b+c$ take $a-2b+3c$.

(2) From $5a^2-17ab-16c^2$ take $4a^2+17ab-15c^2$.

(3) From $\dfrac{3x^2}{2}-5xy-\dfrac{y^2}{5}$ take $x^2-5xy-\dfrac{4y^2}{5}$.

18. Hitherto we have dealt with *numerical* coefficients: the two following examples will show how to add and subtract expressions involving *literal* coefficients.

(1) Add px^3+rx^2-sx to qx^3+rx^2-tx.

$$\begin{array}{l} px^3+rx^2-sx \\ qx^3+rx^2-tx \\ \hline \end{array}$$

answer $(p+q)x^3+2rx^2-(s+t)x$.

For p times x^3 together with q times x^3 is $(p+q)$ times x^3; rx^2 to be added together with rx^2 to be added is $2rx^2$ to be added; sx to be subtracted together with tx to be subtracted is $(s+t)x$ to be subtracted.

Ex. (2) From px^3+rx^2+sx take qx^3+rx^2-tx.

$$\begin{array}{l} px^3+rx^2+sx \\ qx^3+rx^2-tx \\ \hline \end{array}$$

difference $(p-q)x^3+(s+t)x$, by the rule, change the signs and add.

Examples (5).

(1) Add $px+qy+r$ to $qx+py-r$.

(2) Add $3px^2-7qx-r$, $(q-3p)x^2+(7q-p)x-6r$, $7r-qx^2+px$.

(3) From $4x^3-ax^2-3bx$, take $4x^3-4ax^2+2bx$.

(4) From $3(p^2-q^2)x^2-(p^2-3q^2)x-r^3$ take $(2p^2-q^2)x^2+3q^2x+r^3$.

ON BRACKETS.

19. Brackets frequently occur in algebra, and care is needed in their management to avoid error in the signs. The two general rules to be observed are: (1) when the sign + stands before any expression within brackets, the brackets may be removed; thus $a+(b+c)=a+b+c$; and (2) when the sign − stands before an expression in brackets, they may be removed *if the sign of every term within the brackets be changed;*—thus $a-(b-c)=a-b+c$,

where c may stand for any expression enclosed in brackets.

These rules (which follow directly from the rules of addition and subtraction) will enable us to remove *any number* of pairs of brackets which enclose different parts of an expression. Thus, let it be required to simplify $3a-b+\{2b-(a-4b)\}$. Remove the brackets enclosing $2b-(a-4b)$, and by our first rule the expression becomes $3a-b+2b-(a-4b)$, or $3a+b-(a-4b)$. Now remove the bracket enclosing $a-4b$, and we get by our second rule $3a+b-a+4b$, or $2a+5b$. Similarly more complicated expressions may be simplified, and a beginner should remove one pair of brackets at a time, commencing with the outer.

Ex.—$[-\{-(-a)\}]=+\{-(-a)\}$ by the second rule $=-(-a)$ by the first, $=+a$, that is a, by the second.

Examples (6).

Simplify:—

(1) $3a-2b-(2b+3a)$.

(2) $x^2-(y^2-3z^2)-\{2z^2-(2y^2-x^2)\}$.

(3) $-7(a-d)+3(b-c-a)-2[3d-4\{c-2(a-b)\}]$.

(4) $[-\{-(-x)\}]-\{-(-y)\}$.

Negative Quantities.

20. If a man receives a pounds and pays away b pounds, his property is clearly increased by $(a-b)$ pounds. And generally a debt may be regarded as a negative quantity, since it is a quantity to be subtracted from a property, which is positive.

Suppose a man's assets are a and his liabilities b,

what he is really worth is $a-b$. But suppose his debts exceed his assets by c, that is, suppose $b=a+c$; then the expression $a-b$ representing what he is worth is $a-(a+c)$ or $-c$. Hence $-c$ represents a debt of c.

Again, suppose a man walks along a straight road running east and west from a point A to a point B eastwards, and then returns westwards to a point C. If AB be a miles and BC be b miles, then his distance from A, the starting-point (AC), is $a-b$ miles. But suppose b, the distance he walks westwards, is c miles greater than a; then the expression for AC, $a-b$, becomes $a-(a+c)$ or $-c$, and at the same time it is clear he is c miles to the *west* of the starting-point. Hence, in this case, if we consider the distance in one direction *positive*, distance in the opposite direction will be *negative*.

Again, if a man who receives and pays out a number of different sums of money during the day writes down the amounts in a line on a piece of paper, placing + before each sum put into his cash-box, and − before each sum taken out of it, at the end of the day he will ascertain the sum then added to his cash-box by subtracting the − sums from the + sums. But if the former exceed the latter, the result will be negative, and at the same time it is obvious that a loss has been incurred on the day's transactions. Thus the negative result indicates a loss.

These illustrations will enable the reader better to understand the nature of negative quantities, not only as they are used in this elementary treatise, but also in their more extended signification, when the symbols are not confined to arithmetical operations.

CHAPTER III.

Multiplication and Division.

MULTIPLICATION.

21. We know from arithmetic that the product of any two or more numbers is the same, whatever may be the *order* in which we take them: thus, $7\times5=5\times7$. Hence generally, if a and b be any positive numbers, $ab=ba$.

In determining the sign of the product of two *simple* algebraical quantities, there are four cases to be considered :—

(1) $+a\times+b=+ab$. This is a case of common arithmetic.

(2) $-a\times+b=-ab$; thus a debt of a pounds repeated b times is a debt of ab pounds.

(3) $+a\times-b=-ab$; thus a times a debt of b pounds is a debt of ab pounds.

(4) $-a\times-b=+ab$; thus the subtraction of a debt of b pounds is equivalent to an asset of b pounds, and the subtraction of a times a debt of b pounds is an asset of ab pounds.

It will thus be observed that the sign of the product may always be obtained from the signs of its two factors by the law that *like signs produce* $+$, *and unlike signs produce* $-$.

This law may also be thus deduced. Let it be required to find the product of $a-b$ and $c-d$.

$(a-b)(c-d)=(a-b)x$, writing for a moment x for $c-d$.

$$\begin{aligned}(a-b)(c-d)&=ax-bx\\&=a(c-d)-b(c-d)\\&=ac-ad-(bc-bd)\\&=ac-ad-bc+bd\end{aligned}$$

whence we see that $a\times c=ac$, $a\times -d=-ad$, $-b\times c=-bc$, $-b\times -d=bd$.

The reader may work out the following examples in a similar manner.

$$(a+b)(c+d)=ac+ad+bc+bd.$$
$$(a-b)(c+d)=ac+ad-bc-bd.$$
$$(a+b)(c-d)=ac-ad+bc-bd.$$

If any coefficients are numerical, they must be multiplied together and their product be prefixed to the letters; thus

$$4a\times 3b=4\times a\times 3\times b=4\times 3\times a\times b=12ab.$$

A similar process holds for expressions having any number of terms; thus

$$\begin{aligned}(a+b)(c+d-e)&=a(c+d-e)+b(c+d-e)\\&=ac+ad-ae+bc+bd-be\end{aligned}$$

22. The powers of the same quantity are multiplied together *by adding their indices.* For $a^2=a\times a$, $a^3=a\times a\times a$, $\therefore a^2\times a^3=a\times a\times a\times a\times a=a^5$ by definition. And generally, $a^m=a\times a\times a\ldots m$ times; $a^n=a\times a\times a\times\ldots n$ times; $\therefore a^m\times a^n=a\times a\times\ldots m$ times $\times a\times a\times\ldots n$ times, $=a\times a\times\ldots(m+n)$ times $=a^{m+n}$ by definition; m and n being any positive whole numbers.

23. *Examples.*

(1)
$$\begin{array}{r}a+b\\a-b\\\hline a^2+ab\\-ab-b^2\\\hline a^2-b^2.\end{array}$$

(2)
$$\begin{array}{r}a+b\\a+b\\\hline a^2+ab\\ab+b^2\\\hline a^2+2ab+b^2\end{array}$$

(3)
$$\begin{array}{r}a-b\\a-b\\\hline a^2-ab\\-ab+b^2\\\hline a^2-2ab+b^2\end{array}$$

(4)
$$\begin{array}{r}a^2-ab+b^2\\ a+b\\ \hline a^3-a^2b+ab^2\\ a^2b-ab^2+b^3\\ \hline a^3+b^3\end{array}$$

(5)
$$\begin{array}{r}a^2+ab+b^2\\ a-b\\ \hline a^3+a^2b+ab^2\\ -a^2b-ab^2-b^3\\ \hline a^3-b^3\end{array}$$

These five results should be well remembered, viz.:

$(a+b)\ (a-b)=a^2-b^2.$

$(a+b)\ (a+b)$ or $(a+b)^2=a^2+2ab+b^2.$

$(a-b)\ (a-b)$ or $(a-b)^2=a^2-2ab+b^2.$

$(a+b)\ (a^2-ab+b^2)=a^3+b^3.$

$(a-b)\ (a^2+ab+b^2)=a^3-b^3.$

(5)
$$\begin{array}{r}3a+2b\\ 4a+5b\\ \hline 12a^2+8ab\\ +15ab+10b^2\\ \hline 12a^2+23ab+10b^2\end{array}$$

Observe here that the first term of the product is the product of the first terms of the two expressions; the last term is the product of their last terms; and the middle term is the sum of the products of the first term of each expression, and the second term of the other. Thus the product of any two similar binomial expressions may be written down by inspection.

Ex. $(7a+5b)\ (4a+3b)=28a^2+(21ab+20ab)+15b^2$
$=28a^2+41ab+15b^2.$

$$\begin{array}{r}3a+4b\\ 5a-3b\\ \hline 15a^2+20ab\\ -9ab-12b^2\\ \hline 15a^2+11ab-12b^2\end{array}$$

Observe again, the first and last terms of the product are the products of the first and last terms of the two expressions respectively, with the proper signs; and the middle term is the *difference* of the products of the first term in each and the last term in the other, with the proper sign.

Thus $(7a+5b)(4a-3b)=28a^2+(-21ab+20ab)$
$-15b^2=28a^2-ab-15b^2.$

The reader will easily deduce the corresponding rule when the sign of each of the second terms of the binomials is negative :—

Ex. $(7a-5b)(4a-3b)=28a^2-41ab+15b^2$; and thus he will be able to write down at once the product of any binomials whatever.

24. A few more examples will now be worked out.

(1) $(a+b-c)(a+b+c)$

$$\begin{array}{llll}
a+b-c & & & \\
a+b+c & & & \\
\hline
a^2+ab-ac & & & \\
\quad\; ab+b^2-bc & & & \\
\qquad\quad ac+bc-c^2 & & & \\
\hline
a^2+2ab+b^2-c^2 & & &
\end{array}$$

Observe that this result might have at once been written down from the rule that the sum of two numbers multiplied by their difference is equal to the difference of their squares. For $a+b+c$ is the sum $a+b$ and c; $a+b-c$ is the difference of $a+b$ and c; $\therefore$ their product is $(a+b)^2-c^2$ or $a+2ab+b^2-c^2$. There are many similar cases in which a little ingenuity will enable the learner to shorten his work.

(2)

$$\begin{array}{l} x^2-3ax+2a^2 \\ \underline{2x^2+5ax-3a^2} \\ 2x^4-6ax^3+4a^2x^2 \\ \qquad 5ax^3-15a^2x^2+10a^3x \\ \qquad\qquad -3a^2x^2+9a^3x-6a^4 \\ \hline 2x^4-ax^3-14a^2x^2+19a^3x-6a^4 \end{array}$$

In expressions of this kind each should be arranged (if not so given) according to *descending* or *ascending* powers of one of the letters, as this will enable us to collect the partial products more easily. Thus, to multiply $3ax-4a^2+6x^2$ by $2x^2-6a^2-9ax$ we first arrange them thus: $6x^2+3ax-4a^2$ and $2x^2-9ax-6a^2$. In multiplication this rule simplifies our work: in division it will be found to be essential.

(3).

$$\begin{array}{l} 3x-\frac{a}{4} \\ \underline{\frac{x}{2}-\frac{3a}{2}} \\ \frac{3x^2}{2}-\frac{ax}{8} \\ \qquad -\frac{9ax}{2}+\frac{3a^2}{8} \\ \hline \frac{3x^2}{2}-\frac{37ax}{8}+\frac{3a^2}{8} \end{array}$$

The learner should not leave this example with fractional coefficients without clearly understanding the work.

(4)

$$\begin{array}{l} x^2-2px+q^2 \\ \underline{x+3p} \\ x^3-2px^2+q^2x \\ \quad +3px^2-6p^2x+3pq^2 \\ \hline x^3+px^2+(q^2-6p^2)x+3pq^2 \end{array}$$

Observe how the coefficients of x (q^2 and $-6p^2$) are bracketed. The reader will now be able to solve some examples for himself.

Examples (7).

(1) Multiply $3a^2b^3c^4$ by $5a^7bc^3d$; $6a^4b^3$ by $-7ab$; $-m^2a^4b$ by $-\frac{4}{5}m^3a^5$.

(2) Multiply $9a-7b$ by $5a+3b$.

(3) Multiply $3a^2-7ax$ by $5x^2+3a^2$.

(4) Multiply $\frac{a}{3}-\frac{2b}{5}$ by $\frac{3a}{7}-\frac{5b}{6}$.

(5) Multiply $3a^2-2ax+5x^2$ by $4a-3x$.

(6) Multiply $7x^2-4xy-3y^2$ by $3x-4y$.

(7) Multiply $-2xy+4x^2-y^2$ by $3y^2-2xy+4x^2$.

(8) Write down the product of $5a+9b$ and $6a+3b$; and of $3a-7b$ and $2a-9b$.

(9) Write down the products of $a-b+c$ and $a+b-c$, and of $-a+b-c$ and $a+b+c$.

(10) Simplify $9(3x-a)(3x+a)-7(2x+a)^2+3(2x-3a)(x+4a)$.

(11) Find $(a+x)^3$ and $(2x-3a)^3$.

(12) Find the coefficient of x^3 in the product of x^2-ax+p and x^2+bx+q.

[In an example of this kind it is not necessary to find the *whole* product, but only to pick out the terms of it involving x^3.]

(13) Find the square of $9m^2x^2-12mnxy+4n^2y^2$.

(14) Show that the product of $8x^2+2ax-3a^2$ and $10x^2+10ax+5a^2$ may be written $(3x+a)^4-(x+2a)^4$.

What is the value of the product when $x=-2a$?

(15) Prove that $n(n+1)(n+2)(n+3)+1=(n^2+3n+1)^2$.

(16) Prove that $(x^2+yz)^2+(y^2+zx)^2+(z^2+xy)^2-(yz+zx+xy)^2=x^4+y^4+z^4$.

[It is frequently easy to resolve a quadratic expression, such as $2x^2+13ax+15a^2$ into binomial factors, when the expression admits of being so resolved. For since the first and last terms of the expression are respectively the products of the first and last terms of the factors, they may be $(2x+15a)(x+a)$, or $(2x+5a)(x+3a)$, and so on; in general there are only a few possible combinations. Which of these is the proper arrangement is decided by the *middle* term of the expression.

(Read again carefully the notes to section 23.)

A little consideration will show that in the case given, the factors are $(2x+3a)(x+5a)$, the coefficient of middle term being then 13. Practice in examples of this kind will greatly benefit the beginner.]

(17) Resolve into factors: (1), $9x^2-4a^2$; (2), x^2+3x+2; (3), $2x^2+7x+3$; (4), $6x^2+17x+12$.

(18) Resolve into factors: (1), x^4-y^4; (2), x^2-3x+2; (3), $2x^2-5x-3$; (4), $6x^2-x-12$.

(19) Resolve into factors: (1), $16x^2+40xy+25y^2$; (2), $21x^2-41xy+10y^2$; (3), $(4x-7y)^2-9y^2$.

Division.

25. If we divide 7×5 by 5 the result is 7; so $ab\div b=a$.

In dividing 35 by 56, or 7×5 by 7×8, we say $\frac{35}{56}=\frac{5}{8}$. Exactly in the same manner, to divide any positive simple algebraical quantity by another of the same kind, remove any factors common to divisor and dividend, and express the quotient as a fraction in its lowest terms. Thus to divide $abcd$ by $abce$, $\frac{abcd}{abce}=\frac{d}{e}$, removing abc.

26. To divide one power of a quantity by another, subtract the index of the latter from that of the former. Thus $a^5 \div a^2$ is $(a \times a \times a \times a \times a)$ divided by $(a \times a)$, that is $a \times a \times a$, or a^3, that is a^{5-2}. Similarly in any other case.

27. In these cases we have supposed both quantities positive; if one or both be negative, the rule for the sign of the quotient is the same as in the case of multiplication. *Like signs produce* $+$ *and unlike signs produce* $-$ (section 20). For since $-a \times -b = +ab$; $\therefore ab \div -a = -b$, and a similar proof will hold in other cases.

28. If the dividend has more than one term whilst the divisor is a simple quantity only, the quotient is found by dividing *every* term of the dividend by the divisor, affixing the proper sign when the partial quotients are collected together. Thus to divide $abc - abe$ by ab:

$$\frac{abc-abe}{ab} = \frac{abc}{ab} - \frac{abe}{ab} = c - e.$$

Again, to divide $a^3b^2 - a^2b^3$ by a^2b^2:

$$\frac{a^3b^2-a^2b^3}{a^2b^2} = \frac{a^3b^2}{a^2b^2} - \frac{a^2b^3}{a^2b^2} = a - b.$$

Examples (8).

(1) Divide $5abcx$ by ax; $72a^2bc^3$ by $9abc$; $-85a^3b^3$ by $17ab^2$.

(2) Divide $9a^3x^3 - 45a^2x^2$ by $3a^2x^2$; and $65a^5b^2x^4 - 91a^3b^2x^2$ by $-13b^2x^2$.

(3) Divide $(a+b)^3$ by $(a+b)^2$; a^2-b^2 by $a+b$; $a^2 - 2ab + b^2$ by $a-b$; a^3+b^3 by $a+b$; and a^3-b^3 by a^2+ab+b^2.

29. We now come to the more general case, when the divisor has more terms than one. The process of division closely resembles long division in arithmetic,

and the rule is as follows. Arrange dividend and divisor (if not so given) according to either descending or ascending powers of some letter common to the two; divide the first term of the dividend by the first term of the divisor, thus obtaining the first term of the quotient; multiply the divisor by this term and subtract the product from the dividend; bring down such terms of the dividend as may be wanted, and repeat the process till all the terms have been brought down.

The reader will probably see the propriety of this rule after considering two or three examples; it is based on the fact that the divisor is contained in the whole dividend as often as it is contained in all the parts. In fact the principle is the same as it is in arithmetic.

Ex. 1.

$$\begin{array}{l}
a+b)a^2+2ab+b^2(a+b \\
\quad\quad\;\; \underline{a^2+ab} \\
\quad\quad\quad\quad\;\; ab+b^2 \\
\quad\quad\quad\quad\;\; \underline{ab+b^2}
\end{array}$$

Ex. 2.

$$\begin{array}{l}
3a+2b)12a^2-ab-6b^2(4a-3b \\
\quad\quad\quad\;\; \underline{12a^2+8ab} \\
\quad\quad\quad\quad\quad -9ab-6b^2 \\
\quad\quad\quad\quad\quad \underline{-9ab-6b^2}
\end{array}$$

Ex. 3.

$$\begin{array}{l}
2x^2-3xy-y^2)6x^4-17x^3y+11x^2y^2+xy^3-y^4(3x^2-4xy+y^2 \\
\quad\quad\quad\quad\quad\;\; \underline{6x^4-9x^3y-3x^2y^2} \\
\quad\quad\quad\quad\quad\quad\quad -8x^3y+14x^2y^2+xy^3 \\
\quad\quad\quad\quad\quad\quad\quad \underline{-8x^3y+12x^2y^2+4xy^3} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 2x^2y^2-3xy^3-y^4 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{2x^2y^2-3xy^3-y^4}
\end{array}$$

Ex. 4. The reader will easily work out the quotient of a^2+2a+1 divided by $a+1$, viz. $a+1$. But suppose we had not followed the rule to arrange the divisor and dividend *similarly*, and had proceeded thus :

$$
\begin{array}{l}
a+1)1+2a+a^2\left(\frac{1}{a}-\frac{1}{a^2}+\ \&c.\right. \\
\quad\quad\ \ 1+\frac{1}{a} \\
\quad\quad\ \ \overline{-\frac{1}{a}+2a} \\
\quad\quad\ \ -\frac{1}{a}-\frac{1}{a^2} \\
\quad\quad\ \ \overline{\quad\&c.\quad}
\end{array}
$$

and the quotient never terminates. This shows the necessity of *similar* or *symmetrical* arrangement.

Ex. 5. Of course if the divisor will not go exactly into the dividend, there will be a remainder, as in arithmetic. Take the following case :

$$
\begin{array}{r}
x-4)x^2-6x+11(x-2 \\
x^2-4x \\
\hline
-2x+11 \\
-2x+\ \ 8 \\
\hline
3
\end{array}
$$

the remainder is $\frac{3}{x-4}$.

Examples (9).

(1) Divide $12x^2-57x+63$ by $3x-9$ and by $4x-7$.

(2) Divide $30a^2+11ab-28b^2$ by $5a-4b$ and by $6a+7b$.

(3) Divide $24x^3+26x^2-145x-175$ by $2x-5$ and by $3x+7$.

(4) Divide $4x^4y^4+1$ by $2x^2y^2-2xy+1$.

(5) Divide $64-x^6$ by $2-x$.

(6) Divide $x^3-2ax^2+(a^2+b^2)x-ab^2$ by $x-a$.

(7) Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

[Here care is needed in selecting the terms to be brought down; since a is the leading letter, we must bring down the terms in the remainder involving a^2, before the rest, &c.]

(8) Divide 1 by $1-3x$ to four terms.

(9) Divide $\frac{3x^3}{2}-\frac{31x^2y}{12}+\frac{7xy^2}{8}-\frac{9y^3}{16}$ by $\frac{x}{2}-\frac{3y}{4}$.

(10) When is x^2+ax+b exactly divisible by $x+y$?

[Dividing in the ordinary way, there will be found a remainder, $b-ay+y^2$, $\therefore$ the required condition is that $b-ay+y^2=0$.]

CHAPTER IV.

Greatest Common Measure and Least Common Multiple.

30. In arithmetic the greatest common measure of two numbers is the greatest number which will divide both without remainder.

In algebra the greatest common measure of two expressions, arranged according to descending powers of some common letter, is the factor of the highest dimensions, with respect to that letter, which divides both expressions without remainder.

[The highest index of the letter of reference of an expression indicates its dimensions ; thus $3x^2-2x+1$ is of 2 dimensions; x^n-1 is of n dimensions; a^3b^2 or $a \times a \times a \times b \times b$ is of 5 dimensions.]

Thus the meaning of the term *greatest common measure* is somewhat different in algebra from what it is in arithmetic, and *highest common divisor* is sometimes used instead. G. C. M. is for shortness put for greatest common measure; H. C. D. for highest common divisor.

It will be seen, however, that the method of finding the G. C. M. of algebraical quantities closely resembles the arithmetical method.

31. When we can break up the given expressions into their component factors, then G. C. M. can be written down at once.

Ex. (1) The G. C. M. of $9x^2y^4$ and $12x^3y^3$ is plainly $3x^2y^3$.

(2) The G. C. M. of x^5y^2 and $3x^3y^3-2x^4y^2$ is x^3y^2, since the second expression is $x^3y^2(3y-2x)$.

(3) The G. C. M. of two expressions such as x^2-1, $2x^2-5x+3$, may be written down when their factors can be detected; thus $x^2-1=(x-1)(x+1)$, and a little thought will enable the learner to say that $2x^2-5x+3=(x-1)(2x-3)$; hence $x-1$ is the G. C. M. And in a case like this, if the factors of *one* of the expressions be discovered, they will be a guide to find the factors of the other, since presumably there is a factor common to the two.

Before dealing with the general case and giving the rule, some examples similar to the above will be given for solution.

Examples (10).

(1) Find the G. C. M. of $24a^3bc^4$ and $36b^2c^3d^2$.

(2) Of $3ax(a+x)$ and $9x^2(a-x)$.

(3) Of $16a^2b^3$ and $12a^2b-15ab^3$.

(4) Of a^2-b^2 and $a^2-2ab+b^2$.

(5) Of x^3+y^3 and x^2-y^2.

(6) Of $x^2-7x+12$ and $x^2-9x+20$.

(7) Of $6x^2-19x+15$ and $2x^2-11x+12$.

32. If c be a common measure of a and b it will also measure $ma+nb$ and $ma-nb$, where m, n are any quantities.

Thus 3 is a common measure of 15 and 24, and it clearly will measure $15m+24n$ and $15m-24n$, where m and n are any numbers, being contained in the first $(5m+8n)$ times, and in the second $(5m-8n)$ times.

And generally, let c be contained r times in a and s times in b; then $a=rc$, $b=sc$, $ma+nb=mrc+nsc=(mr+ns)c$; hence c measures $ma+nb$.

33. The G. C. M. of any two algebraical quantities may now be thus found. Arrange them according to the powers of some letter, as in division, and divide the one of higher dimensions by the other (if the dimensions be the same, take either expression for dividend). Now, exactly as in arithmetic, take the remainder for new divisor, and the preceding divisor for new dividend, and so on till there is no remainder. Then the last divisor is the G. C. M.

To prove the rule for finding the G. C. M. of two algebraical expressions.

Let A and B represent the two expressions arranged according to descending (or ascending) powers of some common letter, and let B be of not higher dimensions than A. Divide A by B; let p be the quotient, and C the remainder. Divide B by C; let q be the quotient and D the remainder. Divide C by D; let r be the quotient and let there be no remainder; then D is the G. C. M. of A and B.

The process will be thus represented :—

$$
\begin{array}{llll}
\mathrm{B})\,\mathrm{A}\,(p & & & \\
\quad p\mathrm{B} & & & \\
\cline{1-1}
\quad\ \ \mathrm{C})\,\mathrm{B}\,(q & & & \\
\qquad\quad q\mathrm{C} & & & \\
\cline{1-1}
\qquad\quad \mathrm{D})\,\mathrm{C}\,(r & & & \\
\qquad\qquad\ \ r\mathrm{D} & & & \\
\cline{1-1}
\end{array}
$$

from which we have, by the notion of division :—

$$\mathrm{A}=p\mathrm{B}+\mathrm{C}\ (1);\ \mathrm{B}=q\mathrm{C}+\mathrm{D}\ (2);\ \mathrm{C}=r\mathrm{D}\ (3).$$

First we shall show that D is *a* common measure of A and B. By (2) $B=qC+D=qrD+D$ by (3); that is $B=(qr+1)D$ and $\therefore$ D measures B. And in (1) $A=pB+C=pqC+pD+D$ by (2) $=pqrD+pD+D$ by (3) $=(pqr+p+1)D$, $\therefore$ D measures A.

Secondly we shall prove that every common measure of A and B measures D. For let F be any common measure of A and B, and suppose $A=mF$, $B=nF$. Then by (1) * $C=A-pB=mF-npF$; $\therefore$ by (2) $D=B-qC=nF-mF+npF=(n-m+np)F$. Hence F measures D. But no expression higher than D can measure D, $\therefore$ D is the G. C. M.

This is the proof of the arithmetical rule also, which the reader had to take for granted when working sums in G. C. M. in arithmetic.

34. Sometimes there are simple factors in one or both of the expressions whose G. C. M. is required; and in applying the rule just given, one of the remainders at any stage of the process may contain a simple factor; or again the first term of the dividend at any stage may not be exactly divisible by the first term of the divisor. The following rules will enable us to meet these cases.

(1) If the given expressions have any simple factors, the G. C. M. of these factors must be found, and also the G. C. M. of the expressions resulting when these simple factors have been struck out; the latter G. C. M. must be multiplied by the former. Thus the G. C. M. of $12abc(x^2+2ax+a^2)$ and of $18bcd(x^2+3ax+2a^2)$ is $6bc$ (the G. C. M. of $12abc$ and $18bcd$) multiplied by the G. C. M. of $x^2+2ax+a^2$ and $x^2+3ax+2a^2$.

(2) Let the divisor at any step be K and the dividend L, K and L having no simple factors. If the first term

* $A=pB+C$; now if equals be taken from equals the remainders *are equal*, $\therefore$ taking pB away, $A-pB=C$.

of L be not divisible by the first term of K, multiply L by a simple factor l, making it so divisible. Then since K has no simple factor, it has no factor in common with l, and therefore no factor common with Ll but that which it may have in common with L. Hence the result is not affected if at any step we multiply the dividend by a simple factor.

(3) Let, as before, K be the divisor, L the dividend, K and L having no simple factors; let the quotient be m, the remainder R. Suppose R contains a simple factor r, so that R=Sr. Then K, having no simple factor, has no factor in common with r, and therefore the G. C. M. of K and R is the same as that of K and S. Hence the result is not affected, if at any step we remove any simple factor that may occur.

35. *Examples.*

(1) Find the G. C. M. of x^2+2x-3 and x^2+3x-4

$$
\begin{array}{l}
x^2+2x-3)x^2+3x-4(1 \\
\qquad\qquad\quad \underline{x^2+2x-3} \\
\qquad\qquad\qquad\quad x-1)x^2+2x-3(x+3 \\
\qquad\qquad\qquad\qquad\qquad \underline{x^2-x} \\
\qquad\qquad\qquad\qquad\qquad\quad 3x-3 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{3x-3}
\end{array}
$$

$x-1$ is the G. C. M.

(2) Find the G. C. M. of $3x^2-16x-12$ and $5x^3-30x^2-9x+54$.

We must first multiply the second expression by 3, to make the first term divisible by $3x^2$ (see section 34, 2). Thus

$$
\begin{array}{l}
3x^2-16x-12)15x^3-90x^2-27x+162(5x+10 \\
\qquad\qquad\qquad \underline{15x^3-80x^2-60x} \\
\qquad\qquad\qquad\qquad -10x^2+33x+162.
\end{array}
$$

Multiply this by -3, for a similar reason, and it becomes

$$\begin{array}{r} 30x^2 - 99x - 486 \\ 30x^2 - 160x - 120 \\ \hline 61x - 366. \end{array}$$

Divide this remainder by 61 (see section 34, 3), and we get $x-6$ for the new divisor.

$$\begin{array}{l} x-6)3x^2-16x-12(3x+2 \\ \underline{3x^2-18x} \\ 2x-12 \\ \underline{2x-12} \end{array}$$

$x-6$ is the G. C. M.

36. If the G. C. M. of *three* expressions A, B, C be required, we proceed thus: find D, the G. C. M. of A and B, then the G. C. M. of D and C will be the G. C. M. of A, B, C.

For every common measure of A and B measures D; therefore every common measure of A, B, C measures D and C; therefore the highest measure of A, B, C will be the highest measure of C, D.

Least Common Multiple.

37. In arithmetic the least common multiple of two or more numbers is the least number that contains each of them. In algebra the least common multiple of two or more expressions, arranged according to descending (or ascending) powers of some common letter, is the expression of lowest dimensions, with respect to that letter, which is divisible by each of these expressions without remainder.

38. To find the least common multiple (L. C. M.) of two expressions.

Let A and B be the expressions, D their G. C. M., *and let A=PD,* B=QD. It is clear that P and Q have

no common factor (if they had D could not be the *greatest* common measure); therefore the L. C. M. of P, Q, is PQ, and the L. C. M. of PD, QD, is PQD, which may be written $\frac{PD \times QD}{D}$ or $\frac{AB}{D}$. Hence, the L. C. M. of two expressions is their product divided by their G. C. M.

Similarly, if there are three expressions, find the L. C. M. of two of them, and then the L. C. M. of the expression so found and the third; this will be the required L. C. M.

39. In many cases the L. C. M. may be readily found by breaking up each expression into its factors.

Ex. 1. The G. C. M. of $48a^2b^5c^7$ and $120\,b^6c^3d^5$ is easily seen to be $24b^5c^3$; therefore the L. C. M. is

$$48a^2b^5c^7 \times 120b^6c^3d^5 \div 24b^5c^3,$$

that is, $240\,a^2b^6c^7d^5$.

Ex. 2. Find the L. C. M. of x^2-3x+2, x^2-5x+6, and x^2-4x+3.

By inspection, $x^2-3x+2=(x-1)(x-2)$; $x^2-5x+6=(x-2)(x-3)$; $x^2-4x+3=(x-1)(x-3)$. The L. C. M. of the first two is plainly $(x-1)(x-2)(x-3)$, and as this is divisible by the third expression, it is the L. C. M. of the three.

Examples (11).

Find the G. C. M. of—

(1) x^3-2x^2-3x+6 and $x^3-2x^2-6x+12$.

(2) $x^3-5ax^2-7a^2x+35a^3$ and $x^3-5ax^2-4a^2x+20a^3$.

(3) $x^3-3ax^2+3a^2x-2a^3$ and $x^3+ax^2-a^2x+2a^3$.

(4) x^6-x^5-3x+3 and x^6-2x^5-3x+6.

[Try to do this by inspection.]

(5) $2x^3-7x-2$ and $2x^2-x-6$.

(6) $7x^2-23x+6$ and $5x^3-18x^2+11x-6$.

(7) $3x^2-38x+119$ and $x^3-19x^2+119x-245$.

(8) $5a^3+2a^2-15a-6$ and $4a^2-7a^3-12+21a$.

[Remember about *arrangement.*]

(9) $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$.

(10) $7a^3-2a^2b-63ab^2+18b^3$ and $5a^4-3a^3b-43a^2b^2+27ab^3-18b^4$.

(11) $6a^4b-17a^3b^2+14a^2b^3-3ab^4$ and $2a^4+3a^3b-9a^2b^2$.

(12) $x^2-(p+q)x+pq$ and $x^3-2px^2+(p^2+q^2)x-pq^2$.

(13) $x^3-6ax^2+12a^2x-8a^3$, x^4-16a^4, x^3-8a^3.

(14) In this and the following examples find the L. C. M. of the given expressions: $6a^2b^3c^3$ and $15ab^6c^3d$.

(15) $21xy(2x+y)$ and $14y(x+2y)$.

(16) $6(x^2y-xy^2)$ and $16(x^3-y^3)$.

[It is not in practice necessary to find the product of the expressions, and then to divide by the G. C. M.; here, for instance, the G. C. M. is $2(x-y)$: dividing the first expression by this the quotient is $3xy$; this multiplied by the other expression is the L. C. M. For $\frac{AB}{D}=\frac{A}{D}\times B$:—see proof of the rule, section 38.]

(17) $6a^2+13ab+6b^2$ and $9a^2-4b^2$.

(18) a^2-b^2, a^3+b^3, a^3-b^3.

(19) x^2-4, x^3+8, $2x^2+3x-2$, and $2x^2-3x-2$.

(20) $14x^2-5(a-b)x-(a-b)^2$ and $21x^2-(11a+3b)x-2a^2+2ab$.

CHAPTER V.

Fractions.

40. The reader is supposed to know the proofs of the rules in arithmetical fractions, and as our letters and symbols in this course denote the numbers and operations of arithmetic, he will have very little difficulty in understanding the more general expression of these rules, which will be proved to hold, by means of letters which stand for any numbers whatever.

41. Rule for multiplying or dividing a fraction by an integer: $\frac{a}{b} \times c = \frac{ac}{b}$; for in $\frac{a}{b}$ and $\frac{ac}{b}$ the unit is divided into the same number of parts, but in the latter fraction c times as many parts are taken as in the former; hence the latter is c times the former. Similarly $\frac{ac}{b} \div c = \frac{a}{b}$.

Also $\frac{a}{b} \div c = \frac{a}{bc}$; for the same number of parts is taken in each fraction $\frac{a}{b}$, $\frac{a}{bc}$, but in the latter each part is $\frac{1}{c}$th each part in the former, since the unit is divided into c times as many parts as in the former. Similarly

$$\frac{a}{bc} \times c = \frac{a}{b}.$$

42. Since $\frac{b}{b}$ obviously $= 1$, $\therefore$ a may be multiplied by $\frac{b}{b}$ without altering its value, or $a = \frac{ab}{b}$. Hence an integer

may be expressed as a fraction with any denominator by multiplying it by that denominator for the numerator, and by taking that denominator for the denominator.

43. $\frac{ac}{bc}=\frac{a}{b}$. Hence, to reduce a fraction to its lowest terms, divide numerator and denominator by their G. C. M.

44. By the rule of signs in division, $\frac{a}{b}=\frac{-a}{-b}$; $\therefore$ we may change the signs of numerator and denominator of *any fraction* without altering its value.

45. As in arithmetic, fractions are added by finding their least common denominator (that is, the L. C. M. of their denominators), by reducing them to that common denominator, and by adding the numerators, so reduced, for the new numerator, retaining the least common denominator as denominator.

46. Let it be required to multiply one fraction by another, $\frac{a}{b}$ by $\frac{c}{d}$.

Let $\frac{a}{b}=x$, $\frac{c}{d}=y$; then $a=bx$, $c=dy$; $\therefore$ $ac=bdxy$. Now if equals be divided by equals the quotients are equal; divide each of these equals by bd;

$$\therefore \frac{ac}{bd}=xy=\frac{a}{b}\times\frac{c}{d}.$$

Hence the product of the numerators gives the numerator of the result, and the product of the denominators gives its denominator.

47. Lastly, to divide one fraction by another, $\frac{a}{b}$ by $\frac{c}{d}$. As before, let $\frac{a}{b}=x$, $\frac{c}{d}=y$; then $a=bx, c=dy$; $\therefore$ $ad=bdx$, $bc=bdy$; $\therefore$ $\frac{ad}{bc}=\frac{bdx}{bdy}=\frac{x}{y}=\frac{a}{b}\div\frac{c}{d}$.

Hence, we invert the divisor and proceed as in multiplication.

In multiplication and division, it is always well to *cancel* (if practicable), as in arithmetic.

These seven rules will enable us to solve any examples in fractions.

Ex. 1. Reduce $\dfrac{3ab(a^2-b^2)}{9b^2(a^3-b^3)}$ to lowest terms.

The fraction $=\dfrac{3ab(a+b)(a-b)}{9b^2(a-b)(a^2+ab+b^2)}=\dfrac{a(a+b)}{3b(a^2+ab+b^2)}$

by cancelling.

Ex. 2. Add $\dfrac{x-y}{x^2-xy-2y^2}$ to $\dfrac{x+y}{x^2-3xy+2y^2}$.

By inspection, or by applying the rule for G. C. M., we find that the denominators are $(x+y)\,(x-2y)$ and $(x-y)\,(x-2y)$; hence the L. C. D. (least common denominator) is $(x+y)\,(x-y)\,(x-2y)$ and the fractions become

$$\frac{(x-y)^2+(x+y)^2}{(x+y)\,(x-y)\,(x-2y)}$$

which the reader may reduce to

$$\frac{2(x^2+y^2)}{(x^2-y^2)\,(x-2y)}.$$

It is best thus to leave the denominator in factors; no object is gained by multiplying them together.

Ex. 3. From $\dfrac{a^2}{a^3-b^3}$ take $\dfrac{b}{a^2-b^2}$.

The L. C. D. may be easily found to be $(a+b)\,(a^3-b^3)$, and the result of the subtraction is

$$\frac{a^2(a+b)-b(a^2+ab+b^2)}{(a+b)\,(a^3-b^3)}$$

or $$\frac{a^3+a^2b-a^2b-ab^2-b^3}{(a+b)(a^3-b^3)}=\frac{a^3-ab^2-b^3}{(a+b)(a^3-b^3)}.$$

Ex. 4. Multiply $\dfrac{3a^2(a^2-ab+b^2)}{2a+3b}$ by $\dfrac{8ab^2+12b^3}{9a^2(a^3+b^3)}$.

$$\frac{3a^2(a^2-ab+b^2)}{2a+3b}\times\frac{4b^2(2a+3b)}{9a^2(a+b)(a^2-ab+b^2)}$$

$$=\frac{4b^2}{3(a+b)},$$ the other factors cancelling out.

In this and the former example the reader will probably discover the factors without difficulty; but in more complicated cases he can always find the common factors, if any, by applying the rule of G. C. M.

Ex. 5. Divide $\dfrac{x^2+5x+6}{x^2-9}$ by $\dfrac{x^2-x-6}{(x-3)^2}$.

As it is clear $x^2-9=(x+3)(x-3)$, these factors give us a hint in inspecting the numerators, with the view of breaking them up into factors.

Thus $$\frac{x^2+5x+6}{x^2-9}\times\frac{(x-3)^2}{x^2-x-6}$$

$$=\frac{(x+2)(x+3)}{(x-3)(x+3)}\times\frac{(x-3)(x-3)}{(x-3)(x+2)}=1$$

as all the factors cancel out.

Ex. 6. Expressions of various degrees of difficulty and complication are often required to be simplified. Thus: to simplify

$$\left\{\frac{a+b}{2(a-b)}-\frac{a-b}{2(a+b)}+\frac{2b^2}{a^2-b^2}\right\}\frac{a-b}{2b}.$$

Taking first the fractions within the bracket, and

observing that the L. C. D. is $2(a^2-b^2)$, we reduce them to:

$$\frac{a^2+2ab+b^2-(a^2-2ab+b^2)+4b^2}{2(a^2-b^2)}$$

$$=\frac{4ab+4b^2}{2(a^2-b^2)}=\frac{4b(a+b)}{2(a+b)(a-b)}=\frac{2b}{a-b}.$$

Then $$\frac{2b}{a-b}\times\frac{a-b}{2b}=1.$$

Again, to simplify $$\frac{\frac{1}{ab^3}+\frac{1}{a}}{\frac{1}{b}+b-1}$$

This is called a *complex* fraction. The numerator $=\frac{1+b^3}{ab^3}$, and the denominator $=\frac{1+b^2-b}{b}$.

Now $$\frac{1+b^3}{ab^3}\div\frac{1-b+b^2}{b}=\frac{(1+b)(1-b+b^2)}{ab^3}\times\frac{b}{1-b+b^2}$$

$$=\frac{1+b}{ab^2}.$$

Examples (12).

(1) Express $7a^2bc$ as a fraction whose denominator is $4abcd$.

(2) Reduce to lowest terms $\frac{49a^5b^6c^7}{56a^6b^5c^7}$ and $\frac{48x^3yz^3}{54xyz^2}$.

(3) Reduce to lowest terms $\frac{a^3by^2z-ab^2yz^2}{aby^2z^2}$.

(4) Reduce to lowest terms

$$\frac{xy(x^2-y^2)}{9y^3(x^3+y^3)};\quad \frac{a^2-5ab+6b^2}{a^2-4b^2}.$$

(5) Add $\frac{3a-2b}{4b}$ to $\frac{2a-3b}{4a}$, and take $\frac{a^2-ab+b^2}{a-b}$ from $\frac{a^2+ab+b^2}{a+b}$.

(6) Add $\frac{x}{x^2-a^2}$ to $\frac{a}{(x-a)^2}$.

(7) Add $\frac{2}{x}$, $\frac{3}{1-2x}$, $\frac{3-2x}{4x^2-1}$, and subtract the sum of $\frac{1}{2(x+1)}$ and $\frac{1}{x^2}$ from $\frac{1}{2(x-1)}$.

(8) From $\frac{5x-2a}{3x^2-5ax+2a^2}$ take $\frac{2x-5a}{2x^2-5ax+3a^2}$.

(9) Multiply $\frac{65a^4b^3c^2}{68x^3y^4z^3}$ by $\frac{85x^4y^3z^3}{91a^3b^4c^2}$, and divide $\frac{95x^3y^3z^3}{138ab^4c^4}$ by $\frac{114x^4y^3z^3}{161a^3b^3c^3}$.

(10) Multiply $\frac{a^2-x^2}{x+2a}$ by $\frac{x^2+2ax}{a(a-x)^2}$.

Simplify the following expressions:

(11) $\frac{3a-4b}{6}+\frac{2b-5a}{9}-\frac{5a-6b}{12}$.

(12) $\frac{x-y}{xy}-\frac{x-z}{xz}+\frac{y-z}{yz}$. (13) $\frac{28x^2-41x+15}{4x^3-7x^2+7x-3}$.

(14) $\left(\frac{1}{1+x}+\frac{x}{1-x}\right)\div\left(\frac{1}{1-x}-\frac{x}{1+x}\right)$.

(15) $\frac{2a}{(x-2a)^2}-\frac{x-a}{x^2-5ax+6a^2}+\frac{2}{x-3a}$.

(16) $\frac{xy}{ab}+\frac{(x-a)(y-a)}{a(a-b)}+\frac{(x-b)(y-b)}{b(b-a)}$.

(17) $\dfrac{a-b}{(a+c)(b+c)}+\dfrac{b-c}{(a+b)(a+c)}+\dfrac{c-a}{(a+b)(b+c)}$.

(18) $\dfrac{x^2-(b+c)x+bc}{x^2+(b-c)x-bc}$. (19) $\dfrac{2x^2-2xy+x-y}{2x^2+2xy+x+y}$.

(20) $\dfrac{x^2-5x+6}{x^2-5x}\div\dfrac{x^2-3x}{x^2-6x+5}$. (21) $\dfrac{x-1}{1-\dfrac{2}{x+1}}$.

(22) If $\dfrac{a}{b}=\dfrac{c}{d}$, show that $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

[$\dfrac{a}{b}+1=\dfrac{c}{d}+1$, or $\dfrac{a+b}{b}=\dfrac{c+d}{d}$(1), also $\dfrac{a}{b}-1=\dfrac{c}{d}-1$, or $\dfrac{a-b}{b}=\dfrac{c-d}{d}$(2); divide each of the equals in (1) by each in (2) and the quotients will be equal.]

(23) If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, then $\dfrac{a}{b}=\dfrac{ma+nc+pe}{mb+nd+pf}$.

[For let $\dfrac{a}{b}=x=\dfrac{c}{d}=\dfrac{e}{f}$: then $a=bx$, &c.; and $ma=mbx$, $nc=ndx$, $pe=pfx$, $\therefore ma+nc+pe=(mb+nd+pf)x$, &c.]

(24) If $\dfrac{x}{a}+\dfrac{y}{b}=1$, then

$$x^2+y^2+\frac{xy}{ab}(a^2+b^2)=ax+by.$$

[The expression may be written

$$x^2+\frac{axy}{b}+y^2+\frac{bxy}{a}=ax\left(\frac{x}{a}+\frac{y}{b}\right)+by\left(\frac{y}{b}+\frac{x}{a}\right)=ax+by.]$$

CHAPTER VI.

Square Root and Cube Root.

48. The process of multiplying an expression by itself any number of times is termed *involution*. We have already obtained rules for this in the chapter on multiplication, and the reader will hereafter find a general method investigated in the Binomial and Multinomial Theorems.

49. The inverse process of finding an expression which, when multiplied by itself a certain number of times, will produce a given expression, is termed *evolution*, or the extraction of roots.

Thus to extract the *square* root of any expression, is to find an expression which when multiplied by itself becomes equal to the given one. As this is an *inverse* process, the rule must be discovered by observing how the square is obtained.

50. To find the square root of an algebraical expression.

We know that the square of $a+b$ is $a^2+2ab+b^2$.

$$\begin{array}{rl} & a^2+2ab+b^2(a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

Now the square root of a^2, the first term, is a, which is the first term of the root. Subtracting a^2 from

$a^2 + 2ab + b^2$, we then bring down the remainder $2ab+b^2$. Divide this by $2a$ and the result is b, the other term of the root. Lastly, multiply $2a+b$ by b and subtract the product from the remainder $2ab+b^2$.

Suppose there are more terms, and that there is another remainder and another term in the root. The two terms $a+b$ already found must be treated as one, and the square of $a+b$ having by the preceding process been already subtracted, we form a new trial divisor $2(a+b)$ for the next term in the root, exactly as before.

The expression, in order that this inverse process may be successful, must be arranged according to powers (ascending or descending) of some one letter.

51. *Ex.* 1. Find the square root of $9x^2 - 12ax + 4a^2$.

$$
\begin{array}{rl}
 & 9x^2 - 12ax + 4a^2 \ (3x - 2a \\
 & \underline{9x^2 \qquad\qquad} \\
6x - 2a) & \quad -12ax + 4a^2 \\
 & \quad \underline{-12ax + 4a^2}
\end{array}
$$

Ex. 2. Find the square root of

$9a^4 - 12a^3b + 34a^2b^2 - 20ab^3 + 25b^4$.

$$
\begin{array}{rl}
 & 9a^4 - 12a^3b + 34a^2b^2 - 20ab^3 + 25b^4 (3a^2 - 2ab + 5b^2 \\
 & \underline{9a^4 \qquad\qquad\qquad} \\
6a^2 - 2ab) & -12a^3b + 34a^2b^2 \\
 & \underline{-12a^3b + 4a^2b^2} \\
6a^2 - 4ab + 5b^2) & \qquad 30a^2b^2 - 20ab^3 + 25b^4 \\
 & \qquad \underline{30a^2b^2 - 20ab^3 + 25b^4}
\end{array}
$$

Ex. 3. Extract the square root of 6241.

Since the square root of 100 is 10, of 10000 is 100, and so on, the reader will easily see that the square root of any number under 100 has one figure, the square root

of a number under 1000 and more than 100 has two figures, and so on. If then we point off every second figure, beginning with the units, the number of points will give us the number of figures in the square root.

$$\begin{array}{r|l} & \dot{6}2\dot{4}\dot{1}\ (70+9 \\ \hline 140+9 & 1341 \\ (149) & 1341 \\ \hline \end{array}$$

Thus we deduce the arithmetical rule, which the learner has probably already learnt. In practice the cyphers are omitted.

52. Since $(a^2)^2=a^4$, it follows that the *fourth* root of any quantity is the square root of the square root, and may be found by repeating the process of finding its square root.

Examples (13).

(1) Find the square roots of $4x^2y^4z^8$, $81a^6b^{12}$, $\dfrac{9a^4b^6}{16c^2d^2}$, and the fourth root of $\dfrac{625a^{16}b^{20}}{1296x^{12}y^{28}}$.

(2) Find the square root of $25a^2-40ab+16b^2$, of $49c^2-112cd+64d^2$, of $81x^2-198xy+121y^2$.

(3) Of $x^4-2x^3y+3x^2y^2-2xy^3+y^4$.

(4) Of $x^4-6x^3+13x^2-12x+4$.

(5) Of $25a^4-40a^3b+106a^2b^2-72ab^3+81b^4$.

(6) Of $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-2$.

(7) Of $36x^4-36x^3+17x^2-4x+\frac{4}{9}$.

(8) Of $a^2+4ab+6ac+4b^2+12bc+9c^2$.

(9) Of $16a^6-8a^4-16a^3+a^2+4a+4$.

(10) Of $2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{a^4+b^4+c^4}{a^2b^2c^2}$.

[We may either work this as it stands, arranging the terms in the proper order, or we may express it in the form—

$$\frac{a^4+2a^2b^2+2a^2c^2+b^4+2b^2c^2+c^4}{a^2b^2c^2}, \text{ \&c.]}$$

CUBE ROOT.

53. To find the cube root of an algebraical expression.

As in the case of square root, we may deduce the inverse process by observing how $a+b$, the cube root of $a^3+3a^2b+3ab^2+b^3$, may be derived.

$$\begin{array}{rl}
 & a^3+3a^2b+3ab^2+b^3(a+b \\
 & a^3 \\
\hline
3a^2) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
\end{array}$$

Thus, arranging the expression as before, the cube root of the first term is a, the first term of the root; then subtracting a^3 and bringing down the remainder, the second term is obtained by dividing the first term of this remainder by $3a^2$; then subtracting $3a^2b+3ab^2+b^3$ (which in practice may be more easily constructed by multiplying the sum of $3a^2$ and $(3a+b)b$ by b, which is the rule given in arithmetic), we infer that the whole cube root has been extracted, if there is no remainder. If other terms be left, consider $a+b$ (already found) as one term, and the cube of $a+b$ having been already extracted, we form a new trial divisor $3(a+b)^2$, and continue just as before.

54. An example will make the rule clearer. Let it be required to find the cube root of—

$$8x^3-36x^2y+54xy^2-27y^3.$$

$$
\begin{array}{lll}
 & & 8x^3-36x^2y+54xy^2-27y^3(2x-3y \\
 & & 8x^3 \\
\hline
[3a^2=] & 12x^2 \quad) & -36x^2y+54xy^2-27y^3 \\
[(3a+b)b=] & -18xy+9y^2 & -36x^2y+54xy^2-27y^3 \\
\hline
[3a^2+(3a+b)b=] & 12x^2-18xy+9y^2 &
\end{array}
$$

Here $2x$ corresponds to a in the preceding section, $-3y$ corresponds to b.

55. The rule in arithmetic for the extraction of the cube root, which the reader has hitherto taken on trust, is directly deduced from the rule here given.

First, as regards pointing the given number whose cube root is required.

Since $10^3=1000$, $100^3=1000{,}000$, it is clear that the cube root of any number under 1000 consists of one figure; of any number between 1000 and 1000,000, of two figures, and so on. Hence the number of figures in the cube root of any number may be known by pointing off every third figure in it, beginning with the unit place.

To find the cube root of 103823.

$$
\begin{array}{lrl}
 & & 10\dot{3}82\dot{3}\ (40(=a)+7(=b) \\
 & a^3 = & 64000 \\
\hline
3a^2 & =4800 &)\ 39823 \\
(3a+b)b & =\ 889 & 39823=3a^2b+3ab^2+b^3 \\
\hline
3a^2+(3a+b)b=5689 & &
\end{array}
$$

By pointing every third figure we see that there are two figures in the root. The greatest number whose cube is contained in 103000 is $40(=a)$: the trial divisor

and the subtrahend are formed exactly as before. The cyphers (as in the ordinary arithmetical rule) may be omitted in practice.

56. Since $(a^2)^3 = a^6 = (a^3)^2$, the sixth root of an expression may sometimes be found by taking the square root of its cube root, or the cube root of its square root.

Examples (14).

(1) Find the cube root of $a^3 - 6a^2b + 12ab^2 - 8b^3$.

(2) Of $27x^3 + 108x^2y + 144xy^2 + 64y^3$.

(3) Of $125a^3 - 525a^2b + 735ab^2 - 343b^3$.

(4) Of $729x^3 + 2673x^2y + 3267xy^2 + 1331y^3$.

(5) Of $8a^6x^6 - 84a^4x^4by^3 + 294a^2x^2b^2y^6 - 343b^3y^9$.

(6) Of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

(7) Find also the sixth root of the last expression.

(8) Find the cube root of 24389 and of 970299.

(9) Find the sixth root of 2985984 and of 4826809.

(10) Find the cube root of
$$64a^6 - 48a^5 - 84a^4 + 47a^3 + 42a^2 - 12a - 8.$$

CHAPTER VII.

Simple Equations involving one unknown quantity.

57. We have had numerous examples of the equality of two algebraical expressions :—for instance

$$a(x+a)=ax+a^2; \ (x+a)^2=x^2+2ax+a^2.$$

These are, respectively, said to be *identical* expressions; that is $a(x+a)=ax+a^2$ is true whatever be the value of a and x; so with the identity $(x+a)^2=x^2+2ax+a^2$. But suppose we have such a relation as $x+3=7$, we see at once that this is not always true, but only when x has the value 4. A relation which holds true for one value (or for certain values) only of the letter (or letters) in it is called an equation; which is thus an *algebraical sentence*, stating the equality of two expressions. It is customary to employ the last letters of the alphabet to denote the particular values which cause the relation between the quantities to hold true, and when an equation is proposed for solution, what we have to do is to find these particular values which *satisfy* it. These values are called the *roots* of the equation.

58. By a simple equation of one unknown quantity is meant an equation in which only the *first* power of the unknown quantity is found, and in which there is only one unknown quantity. The rules for solving an equation of this kind are extremely simple; they depend on the following principles.

59. (1) In any equation any quantity may be transferred from one side to the other without affecting the result, the sign of the quantity so transferred being changed.

For let the equation be $x-a=b$. Now if equals be added to equals the wholes are equals: add a to each side and $x=a+b$. Again, if equals be taken from equals, the remainders are equals: take b from each side and $x-a-b=0$. If we took a more complicated case the reasoning would be just the same. This process is called transposition.

(2) If every term on each side be multiplied (or divided) by the same quantity, the results will be equal.

This is obvious: thus if $\frac{x}{2}=4$, multiplying each side by 2, $x=8$; if $5x=25$, dividing each side by 5, $x=5$.

60. To solve any simple equation of one unknown quantity, first *clear* it of fractions (if any) by multiplying each side by their least common denominator (59,2); then transpose the terms involving the unknown quantity to one side, and the other terms to the other side (by 59,1); lastly divide the latter side by the coefficient of the unknown quantity (59,2), and the result is the root.

61. Examples.

(1) $3x-6=x+4$.

$3x-x=6+4$, or $2x=10$, and $x=5$.

(2) $4-\frac{2x}{3}=\frac{9}{2}-x$.

$24-4x=27-6x$;

$6x-4x=27-24$, or $2x=3$, and $x=\frac{3}{2}$.

(3) $\frac{x-6}{7}-\frac{10-x}{2}=\frac{x}{14}-1\frac{2}{7}$.

$2x-12-7(10-x)=x-18$,

or $9x-82=x-18 \therefore 8x=64$, and $x=8$.

(4) $\frac{x-a}{b}=\frac{x-b}{a}$.

$ax-a^2=bx-b^2$;

$x(a-b)=a^2-b^2 \therefore x=a+b$.

(5) $\frac{b^2}{x}+c=\frac{a^2}{c}$.

$b^2c+c^2x=a^2x$;

$x(c^2-a^2)=-b^2c \therefore x=\frac{b^2c}{a^2-c^2}$.

(6) $\frac{7+11x}{51}-\frac{12x-21}{27}-3\frac{1}{17}=\frac{8}{153}-\frac{8x-4}{9}$.

L. C. D. is 459 and

$$9(7+11x)-17(12x-21)-1404=24-51(8x-4),$$

or $63+99x-204x+357-1404=24-408x+204$;

$$(408+99-204)x=1404-63-357+24+204;$$

$$303x=1212, \text{ and } x=4.$$

When there are decimals in an equation, it is generally best to replace them by the equivalent vulgar fractions. As the learner becomes more expert in handling algebraical expressions, he will be able to shorten the work by various artifices adapted to the particular equation. Some examples of the ordinary forms of simple equations will now be given for solution.

Examples (15).

Solve the following equations—

(1) $3x-7=13+x$; and $27x+5=40-8x$.

(2) $-8x+15=-11x-6$;
and $-3x-(7-2x)=21-(4x+13)$.

(3) $4x-3\frac{1}{2}=6\frac{1}{3}-55x$;
and $\frac{3x}{2}-\frac{5}{7}=21x-\left(\frac{2x}{3}+3\frac{17}{42}\right)$.

(4) $\frac{5x}{7}-\frac{2}{3}-\frac{4x}{3}=\frac{3x}{14}-4$; and $\frac{3x}{2a}-\frac{4x}{3a}=1-\frac{x}{a}$.

(5) $\frac{ax-b^2}{b}=\frac{bx-a^2}{a}$; and $\frac{ax}{b}-\frac{c}{d}=\frac{b}{a}-\frac{dx}{c}$.

(6) $\frac{2m^2}{3x}-\frac{3n^2}{2x}=\frac{m}{3}-\frac{n}{2}$; and $\frac{(m+n)(n-x)}{m-n}+x=0$.

(7) $\frac{3x-7}{4x+2}=\frac{3x-14}{4x-13}$; and $\frac{1}{x-1}-\frac{2}{x+7}=\frac{1}{7(x-1)}$.

(8) $x(x-5)+3(1-x^2)=8-2x^2$; and
$(x-3)^2+(x-1)(x-2)=3x-2x(5-x)$.

(9) $\frac{12}{x}+\frac{1}{12x}=\frac{29}{24}$; and $\frac{1}{6-x}-\frac{1}{7-x}=\frac{1}{x^2}$.

(10) $\frac{x+1}{6}+\frac{3x-1}{8}-\frac{5x-7}{12}+1=\frac{7x-5}{24}$.

(11) $\frac{x}{4}-\frac{x+10}{5}+4\frac{3}{4}=x-1-\frac{x-2}{3}$.

(12) $\frac{8}{35}(8x+3)-\frac{7}{15}(7x-13)=\frac{1}{21}(47x-167)$.

(13) $x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}(2x-57)-\frac{5}{3}$.

(14) $-\cdot 1x+\frac{x-7\cdot 655}{\cdot 23}=\cdot 7$.

(15) $\frac{7\cdot 1-3x}{\cdot 5}-\cdot 8x=9\cdot 44$.

(16) $\frac{3x+1}{4}=\frac{9x+8}{12}-\frac{x+1}{x+8}$.

[Before clearing off fractions, write the equation thus:

$$\frac{3x}{4}+\frac{1}{4}=\frac{3x}{4}+\frac{2}{3}-\frac{x+1}{x+8}\ \therefore\ \frac{5}{12}=\frac{x+1}{x+8},\ \text{\&c.}]$$

(17) $\frac{7x+1}{x-1}=\frac{35}{9}\left(\frac{x+4}{x+2}\right)+\frac{28}{9}$.

(18) $\frac{471-6x}{2}-\frac{402-3x}{12}=9-\frac{2x+1}{29}.$

(19) $\frac{6x+7}{9x+6}-\frac{5x-5}{12x+8}=\frac{1}{12}.$

(20) $(a+x)(b+x)-a(b+x)=\frac{a^2c}{b}+x^2.$

[The first side is $(b+x)(a+x-a)$ or $bx+x^2$, hence striking out x^2, at once $x=\frac{a^2c}{b^2}.$]

(21) $\frac{25-\frac{x}{3}}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=\frac{23}{x+1}+5.$

[In a case like this, simplify before clearing the fractions.]

(22) $\frac{5x-94}{65}-\frac{7x+157}{52}+3\frac{1}{20}=\frac{4x+164}{39}-11\frac{11}{20}.$

(23) $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{3x+1}{21}-\frac{2x-2\frac{1}{3}}{6}+\frac{1}{105}.$

(24) $3+\frac{1-\frac{x}{2}}{4-\frac{3}{2-x}}-\frac{x-\frac{x}{1+x}}{10-\frac{18x}{x+1}}=0.$

Problems which may be solved by means of simple equations with one unknown quantity.

62. A great variety of questions, at first sight complicated and even insolvable, are made very simple by expressing them in algebraical language, in the form of a sentence or equation. In such questions there are certain things given, from which some unknown element in the problem is to be discovered. If we write x for this unknown quantity and express the data in algebraical

language in the form of an equation, its solution will determine this unknown element. The following examples will give the reader some practice in problems of different kinds, each kind representing a large class of questions.

(1) Divide 68 into two parts so that one may be three times the other.

Let x be the smaller part, then $3x$ is the other, and $x+3x=68$, whence $x=17$, and the other part, $3x=51$.

(2) The ages of two men are now as 4 to 3, but 9 years ago they were as 3 to 2; find their present ages.

Let x = present age of the elder, then $\frac{3x}{4}$ is the age of the other. Now 9 years ago their ages were $x-9$ and $\frac{3x}{4}-9$, hence by the question $\frac{3x}{4}-9=\frac{2}{3}(x-9)$, which solved gives $x=36$, the age of one $\therefore \frac{3x}{4}=27$, the age of the other.

(3) How much wine at 10*s.* per gallon must be mixed with 44 gallons at 15*s.* per gallon, so that the mixture may be worth 12*s.* 9*d.* per gallon?

Let x=no. of gallons to be mixed; then $10x$=their value in shillings, and the value of 44 at 15*s.* per gallon is 660 shillings; $\therefore$ the value of the mixture is $10x+660$, and the no. of gallons in it is $x+44$; $\therefore$ the price of each gallon is $\frac{10x+660}{x+44}$. But this is 12*s.* 9*d.* or $12\frac{3}{4}$ shillings. Hence $\frac{10x+660}{x+44}=12\frac{3}{4}$, which gives, when solved, $x=36$.

(4) How may a debt of £5 be paid with 29 coins, of which some are crowns, the rest florins?

Let x=no. crowns $\therefore$ their value is $5x$. The no.

florins is $29-x$ $\therefore$ their value is $2(29-x)$ in shillings. But the debt is 100 shillings; $\therefore$ $5x+2(29-x)=100$; whence $x=14$, and $29-x=15$.

(5) A and B have two guineas between them, and if A give to B one shilling for every penny B has, A will then have ten shillings less than B: how much had each at first?

Let x = what A had in shillings $\therefore$ $42-x$=what B had in shillings, and $12(42-x)$=the no. of pence B had $\therefore$ A is to give $12(42-x)$ *shillings*. A will then have $x-12(42-x)$ and B $42-x+12(42-x)$ $\therefore$ by the question $x-12(42-x)+10=42-x+12(42-x)$, from which the reader will find $x=40$. Hence A had £2, B had 2*s*.

(6) A and B walk over the same ground going out one way and coming home in the other, starting at the same time in opposite directions. A walks $3\frac{3}{4}$ miles per hour and B 4. A wants $\frac{1}{4}$ of a mile of being half-way when he meets B. Required the length of the walk.

Let it be x miles; then when they meet, A has walked $\frac{x}{2}-\frac{1}{4}$, and since he walks $3\frac{3}{4}$ in one hour, he will walk $\frac{x}{2}-\frac{1}{4}$ in $\left(\frac{x}{2}-\frac{1}{4}\right)\div 3\frac{3}{4}$ hours. Similarly B has walked $\frac{x}{2}+\frac{1}{4}$, and the time he takes to do this at 4 miles per hour is $\left(\frac{x}{2}+\frac{1}{4}\right)\div 4$ hours. But these times are equal;

$$\therefore \frac{2x-1}{4}\div 3\tfrac{3}{4}=\frac{2x+1}{4}\div 4.$$

The reader will find by solving this equation $x=15\frac{1}{2}$ miles, the length of the walk.

Examples (16).

(1) Divide 12 into two parts, such that one of them is seven times the other.

(2) Find the number to which if its seventh part be added, the sum will exceed its half by 27.

(3) A sum of £7 10*s.* is made up of shillings and half-crowns; there are five times as many shillings as half-crowns: find the number of each.

(4) The prices of admission to a concert were half-a-crown for adults, eighteenpence for children (who composed a quarter of the whole audience). If £18 were taken, find the number of children.

(5) A man invests half his property at 5 per cent., a third at $2\frac{1}{2}$ per cent., and the remainder at 4 per cent. What is his property if his income is £360?

(6) A sum of money is divided among three persons. The first receives £10 more than a third of the whole sum; the second receives £15 more than a half of what remains; and the third receives what is over, which is £70. Find the original sum.

(7) How much tea at 2*s.* 6*d.* per lb. must be added to 72 lbs. at 3*s.* 4*d.* per lb., so that the mixture may be worth 3*s.* $1\frac{1}{2}$*d.* per lb.?

(8) A merchant mixes wine at 12*s.* per gallon with 40 gallons at 15*s.* per gallon, and sells the mixture at £2 2*s.* per dozen (2 gallons), thus gaining 50 per cent. How much does he mix?

(9) In a walking match of 12 miles, one boy has a start of 20 minutes, and walks at the rate of $4\frac{1}{2}$ miles per hour. What is the rate of the other boy if the result is a dead heat? What is his rate if the boy who has the start wins by half a mile?

[A question of this kind can be done by common arithmetic.]

(10) In an examination paper one boy, A, gets a marks more than the mth part of full marks, while another, B, gets b marks less than the nth part of full marks. The marks obtained by A and B are in the ratio of a to b. Find the full marks.

(11) A boy is three times as old as his brother; in four years he will be twice as old: what are their ages?

(12) In a division the majority was 140, which was two-thirteenths of the whole number of voters; how many voted on each side?

(13) A and B begin to play with equal sums, and when B has lost five-elevenths of what he had at starting, A has gained £6 more than half what B has left; what had they at first?

(14) A ship 40 miles from the shore springs a leak which admits $3\frac{3}{4}$ tons of water in 12 minutes; 60 tons would sink her, but the pumps can throw out 12 tons in an hour. At what rate must she sail so as to reach the shore just as she begins to sink?

(15) A and B ride over the same ground but in opposite directions, and B starts 10 minutes after A, but rides twice as fast. They meet half-way, which is two miles from the starting point. Find the rate of each.

(16) At what time between n and $n+1$ o'clock is the minute hand m minutes in advance of the hour hand?

[Let x = no. of minutes advanced by the minute hand since n o'clock; then the number of minutes advanced by the hour hand is $x-m-5n$, $\therefore$ since the minute hand travels twelve times as fast as the hour hand, $x=12(x-m-5n)$, whence $x=\frac{12}{11}(m+5n)$. The reader will easily follow this reasoning by drawing a diagram: it is the general case of all problems of this kind.]

(17) When will the hands be together between one and two o'clock?

(18) When will they be at right angles between two and three?

(19) When will they be just opposite between three and four?

(20) In the same time, A can do twice as much work, B one and a half times as much work, as C. The three work together for two days, and then A works on alone for half a day. In what time could A and C together do as much as the three will have thus performed?

[Let $x=$ time in which C would do this work by himself. Call the work W for shortness, then in one day C would do $\frac{W}{x}$; $\therefore$ by the question, in one day A would do $\frac{2W}{x}$, and B $\frac{3W}{2x}$. Hence the three together would in one day do $\frac{W}{x}+\frac{2W}{x}+\frac{3W}{2x}=\frac{9W}{2x}$, and in two days $\frac{9W}{x}$. Now in half a day A does $\frac{W}{x}$, $\therefore$ at the end of $2\frac{1}{2}$ days, working as described in the question, $\frac{9W}{x}+\frac{W}{x}$ or $\frac{10W}{x}$ will be done. Again in one day A does $\frac{2W}{x}$, and C does $\frac{W}{x}$, $\therefore$ in one day A and C working together would do $\frac{2W}{x}+\frac{W}{x}$, or $\frac{3W}{x}$. The question now is, in what time will $\frac{10W}{x}$ be done, if $\frac{3W}{x}$ be done in one day? Plainly in $\frac{10W}{x}\div\frac{3W}{x}$ days, that is in $\frac{10}{3}$ or $3\frac{1}{3}$ days. This is a fairly general case of questions involving different rates of work, different rates of supplying a cistern by two or more taps, &c.]

(21) A cistern is supplied from three taps: by the first it would be filled in 48 minutes, by the second in 60, by the third in 72; in what time would it be filled if all run together?

(22) A can do a piece of work in 8 days and B can

do it in 10 days; A works alone for 6 days, if B then helps him, when will the work be done?

(23) How many minutes does it want to four o'clock, if three quarters of an hour ago it was twice as many minutes past two?

(24) A woman took eggs to market and sold first six less than half; then six less than one-third of the remainder; finally six less than one-fourth of those then remaining. On counting her eggs she found she had sold half. How many had she at first?

(25) I can examine 12 sets of papers which are arranged in order in the same time as 7 sets not so arranged. I examined 490 when 80 per cent. were properly arranged; how many could I have examined in the same time if they had been all arranged in order?

(26) One clock gains 3 minutes in 7 days, and a second loses $\frac{2}{3}$ minute in 5 days; if they are set to the right time, when will they next be together? And when will they next both show right time?

(27) A basket of oranges is emptied by one person taking half of them and one more, a second person taking half the remainder and one more, and a third person taking half the remainder and six more. How many did the basket contain at first?

(28) One cwt. of bronze contains by weight 70 per cent. of copper, and 30 per cent. of tin; with how much copper must it be melted in order to contain 84 per cent. of copper?

(29) A farmer has a number of hurdles, 6 ft. long; he finds that by arranging them so as to enclose a given space of ground, if he place them 1 ft. distant from each other, he has not enough by 80; but if he place them a yard apart, he has 50 hurdles to spare; how many hurdles has he?

(30) A and B start to run a race; at the end of 5 minutes, when A has run 900 yards, and is 75 yards ahead, he slackens his pace by 20 yards a minute, and thus reaches the winning post half a minute behind B, who runs uniformly throughout. How long did the race last?

CHAPTER VIII.

Simple Equations involving more than one unknown quantity.

63. When there are *two* independent simple equations involving two unknown quantities, they may always be reduced to one equation involving one unknown quantity.

Thus, suppose the equations are $x-y=3$, $3x+5y=25$; from the first $x=3+y$, and *substituting* this value of x in the second, it becomes $3(3+y)+5y=25$; whence $y=2$, and $x=3+y=5$.

Equations of this kind are called *simultaneous* equations.

64. Observe, the equations must be *independent.* Two such equations as $2x-2y=6$, $3x-3y=9$ are not independent, for dividing each side of the first by 2, and each side of the second by 3, we see that the relation expressed is the same in each equation, viz. $x-y=3$. Thus they represent one and the same equation.

65. Simultaneous equations of one dimension, involving two unknown quantities, can thus always be solved by substituting in one of them the value of one of the unknown quantities derived from the other. This process is called elimination.

66. In practice this substitution may be made in different ways, more or less suitable in any particular case.

Ex. 1. Thus, to solve $4x-7y=6$, $5x-6y=13$. From the first, $x=\frac{1}{4}(7y+6)$, $\therefore$ substituting this in the second, $\frac{5}{4}(7y+6)-6y=13$, or $35y+30-24y=52$, whence $11y=22$ and $y=2$ $\therefore$ $x=\frac{1}{4}(7y+6)=\frac{20}{4}=5$. This is *direct* substitution.

Again, from the first, $x=\frac{7y+6}{4}$, and from the second $x=\frac{6y+13}{5}$; whence equating these values,

$\frac{7y+6}{4}=\frac{6y+13}{5}$, $\therefore$ $35y+30=24y+52$, and $11y=22$ $\therefore y=2$, and $x=\frac{7y+6}{4}=5$ as before. This is sometimes called the method of comparison.

Thirdly, if we multiply the first by 5 we get $20x-35y=30$, and if we multiply the second by 4, $20x-24y=52$. Now if equals be subtracted from equals the remainders are equals :—

$$\begin{array}{lr} & 20x-35y=30 \\ & 20x-24y=52 \\ \hline \text{(subtracting)} & -11y=-22 \text{ or } y=2, \end{array}$$

and then $x=5$ as before.

This third method is sometimes called cross multiplication. We multiply the given equations (first reduced, if necessary), by such quantities as will equalise the coefficients of one of the unknown quantities, and then *eliminate* (i.e. get rid of) that unknown quantity by addition or subtraction. In most cases this method is the simplest and best.

Ex. 2. $$\left.\begin{array}{r} 2x+35y=76 \\ 12x-y=34 \end{array}\right\}$$

Multiply the first by 6 and subtract the second from it; thus—

$$\begin{array}{r} 12x+210y=456 \\ 12x \quad -y=\ 34 \\ \hline 211y=422 \end{array}$$

and $y=2$; whence $12x=y+34=36$, and $x=3$.

The learner should solve the first ten of the following examples by each of the methods which have been explained.

Examples (17).

(1) $\left.\begin{array}{l} x-3y=2 \\ y-3x=-22 \end{array}\right\}.$ (2) $\left.\begin{array}{l} 5x-4y=19 \\ 3x+y=25 \end{array}\right\}.$

(3) $\left.\begin{array}{l} 9x-8y=44 \\ 11x+30y=14 \end{array}\right\}.$ (4) $\left.\begin{array}{l} 7x-16y=42 \\ 5x+17y=30 \end{array}\right\}.$

(5) $\left.\begin{array}{l} 9x-23y=-43 \\ 17x+11y=191 \end{array}\right\}.$ (6) $\left.\begin{array}{l} 10x-9y=22 \\ 13x+20y=1043 \end{array}\right\}.$

(7) $\left.\begin{array}{l} 2x+\cdot 4y=1\cdot 2 \\ 3\cdot 4x-\cdot 02y=\cdot 01 \end{array}\right\}.$ (8) $\left.\begin{array}{l} 2x-3y=1 \\ \dfrac{x+y-6}{7}=\dfrac{x-y+1}{4} \end{array}\right\}.$

[The second equation in ex. 8 must first be expressed in its simplest form. Similarly in all other cases of the same kind.]

(9) $\left.\begin{array}{l} 2(x+y)=3(x-y)+10 \\ 2x-y=4(2y-x)+3 \end{array}\right\}.$

(10) $\left.\begin{array}{l} \dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5} \\ 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3} \end{array}\right\}.$

(11) $\left.\begin{array}{l} \dfrac{x+y}{3}+\dfrac{y-x}{2}=9 \\ \dfrac{x}{2}+\dfrac{x+y}{9}=5 \end{array}\right\}$ (12) $\dfrac{5x}{6}-\dfrac{3y}{7}=2=\dfrac{x}{2}-\dfrac{y}{7}.$

(13) $\left.\begin{array}{l} ax=by \\ \dfrac{x}{a-b}+\dfrac{y}{a+b}=\dfrac{a^2+b^2}{a^2-b^2} \end{array}\right\}.$ (14) $\left.\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b}=m \\ \dfrac{x}{c}+\dfrac{y}{d}=n \end{array}\right\}.$

(15) $ax+by=\frac{1}{2}(bx+ay)=a^2-b^2$.

(16) $\left.\begin{array}{l}\frac{2}{x}+\frac{3}{y}=13\\ \frac{3}{x}-\frac{4}{y}=-6\end{array}\right\}$.

[Multiply the first by 3, the second by 2, and subtract; then $\frac{17}{y}=51$, whence $y=\frac{1}{3}$, &c.]

(17) $\left.\begin{array}{l}3x+\ \ 5y=8xy\\ 9x-10y=\ \ xy\end{array}\right\}$.

[Divide each by xy, &c.]

(18) $\left.\begin{array}{l}9x+8y=43xy\\ 8x+9y=42xy\end{array}\right\}$.

(19) $\left.\begin{array}{l}2y-\frac{3-x}{7}=\frac{20-y}{4}-x\\ \frac{3x+y}{25}+\frac{3y-x}{35}+x+11=0\end{array}\right\}$.

(20) $\frac{x-a}{b}+\frac{y-b}{a}=\frac{x}{a}+\frac{y}{b}=1$.

(21) $\left.\begin{array}{l}axy=c(bx+ay)\\ bxy=c(ax-by)\end{array}\right\}$.

(22) $ay+bx-cxy=ax-cy=0$.

67*. If there be three independent simple equations containing three unknown quantities, we can deduce from two of them an equation in which one of the unknown quantities has been eliminated, and then from the

* Pupil-teachers are not at present required to solve equations with more than two unknown quantities, but they should not omit this easy section, which will give them practice in eliminations.

third and either of the two others we can deduce another equation by eliminating the same unknown quantity. We shall then have two equations involving two unknown quantities which may be solved as before.

Ex. 1. $$\left.\begin{aligned} 3x-2y-z&=4 \\ x+3y-2z&=9 \\ 2x-y+3z&=11 \end{aligned}\right\}.$$

(1) Multiply the second by 3 and subtract the first from it: this will give $11y-5z=23$.

(2) Multiply the second by 2 and subtract the third from it, and $7y-7z=7$, or $y-z=1$.

(3) Solve the equations thus obtained, and we find $z=2$, $y=3$; whence x may be found from any one of the three given equations.

Ex. 2. $$\left.\begin{aligned} \frac{2a}{x}-\frac{b}{y}+\frac{3c}{z}&=9 \\ \frac{a}{x}+\frac{b}{2y}+\frac{c}{3z}&=3 \\ \frac{3a}{x}+\frac{2b}{y}-\frac{c}{z}&=4 \end{aligned}\right\}.$$

(1) Subtract the first from twice the second, and $\frac{2b}{y}-\frac{7c}{3z}=-3$. (2) Subtract the third from thrice the second, and $-\frac{b}{2y}+\frac{2c}{z}=5$. (3) We now have obtained two equations from which x has been eliminated; add the first of them to four times the other, and $\frac{17c}{3z}=17$, whence $z=\frac{c}{3}$: then from either of the two $y=\frac{b}{2}$, and from any one of the original equations $x=a$, by substitution.

Examples (18).

(1) $\left.\begin{array}{l} x+y+z=5 \\ x+y=z-7 \\ x-3=y+z \end{array}\right\}.$

(2) $\left.\begin{array}{l} x+y=28 \\ y+z=22 \\ x+z=26 \end{array}\right\}.$

(3) $\left.\begin{array}{l} 3x+5y-4z=25 \\ 5x-2y+3z=46 \\ 3y+5z-x=62 \end{array}\right\}.$

(4) $\left.\begin{array}{l} 7y+8z=67 \\ 9z-5x=21 \\ 12x-5y=11 \end{array}\right\}.$

(5) $\left.\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12 \\ \frac{x}{2}+\frac{z}{3}=10 \\ \frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8 \end{array}\right\}.$

(6) $\left.\begin{array}{l} x+y+z=a+b+c \\ x+a=y+b=z+c \end{array}\right\}.$

(7) $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{z}=\frac{4}{3} \\ \frac{1}{y}+\frac{2}{x}=\frac{3}{2} \\ \frac{3}{z}-\frac{2}{y}=2 \end{array}\right\}.$

(8) $\left.\begin{array}{l} \frac{1}{7y}-\frac{3}{5z}+\frac{5}{8x}=\frac{21}{280} \\ \frac{3}{y}+\frac{17}{70}=\frac{4}{x}-\frac{5}{z} \\ \frac{4}{z}+\frac{6}{x}+\frac{5}{y}=\frac{5}{4} \end{array}\right\}.$

Problems which lead to Simple Equations with more than one unknown quantity.

68. We have seen that a great variety of problems may be solved by expressing their conditions in algebraical language, and by then discovering the unknown element of the question from the solution of the algebraical sentence (or equation) so formed. Conversely every equation, or set of simultaneous equations, may be translated into ordinary language in the form of a problem. Thus the equation $x-44=\frac{x}{4}+\frac{x}{5}$, so interpreted, is :—What number is that whose fourth and fifth parts are to-

gether less than it is by 44? Again, the simultaneous equations $x+y=7$, $\frac{x}{3}+\frac{y}{4}=1$, may be read :—Find two numbers whose sum is 7, and the third part of the one with the fourth of the other is 1. So with other more complicated cases.

69. In expressing a problem in algebraical language, it is sometimes convenient (or it may be necessary) to write down two or more algebraical sentences in which there are two or more unknown elements, these equations being deduced from the conditions of the problem.

Ex. 1. A bill of 25 guineas is paid with crowns and half-guineas, and twice the number of half-guineas exceeds three times that of the crowns by 17; how many are there of each?

Let x=number of half-guineas, y=number of crowns. Then *value* of the x half-guineas is $\frac{21x}{2}$, and of the y crowns $5y$, in shillings, $\therefore$ by first condition of question $\frac{21x}{2}+5y=525$. (1) And by second condition $2x-3y=17$ (2). By solving these equations, $x=40$, $y=21$.

[In problems of this kind, which are very numerous, always express each value in the *same* denomination—shillings in this case.]

Ex. 2. If 1 be added to the numerator of a certain fraction it becomes $\frac{1}{3}$, but if 1 be added to its denominator it becomes $\frac{1}{5}$. Find the fraction.

Let it be $\frac{x}{y}$, then the two conditions, algebraically written, are $\frac{x+1}{y}=\frac{1}{3}$; $\frac{x}{y+1}=\frac{1}{5}$. Whence we shall easily find $x=2$, $y=9$, and the fraction is $\frac{2}{9}$.

Ex. 3. A number consists of three digits, the right-hand one being cipher. If the left-hand and middle digits be interchanged, the number is diminished by 180; if the left-hand digit be halved, and the middle and right-hand digit be interchanged, the number is diminished by 336: find the number.

[In 245 the digits are 2, 4, 5; and 245 represents 200 + 40 + 5.]

Let x be the left-hand, y the middle digit: it is given that the right-hand is 0. Then the number is $100x+10y$, and if the digits be interchanged as in the first supposition, the number would be $100y+10x$. Hence, writing the first relation algebraically: $100y+10x=100x+10y-180$, or reducing: $x-y=2$ . . . (1).

Secondly, if the left-hand digit be halved, and 0 be in the middle, y on the right-hand, the number would be $100.\frac{x}{2}+0+y$ or $50x+y$. Hence, by second hypothesis:—

$50x+y=100x+10y-336$, or reducing:—

$50x+9y=336$ . . . (2).

By (1) and (2) we shall find $x=6$, $y=4$, and the number is 640.

This is an example of a large class of questions.

Ex. 4. The rent of a farm consists of a certain fixed money payment, together with the value of a certain number of quarters of wheat. When wheat is at 50*s.* per quarter, the whole rent amounts to £625; when wheat is at 56*s.*, the rent is £685: find the fixed money payment.

Let it be x pounds, and let y=the number of quarters. Then, by first hypothesis:—

$$x+\frac{5y}{2}=625.$$

[Observe we must express the second term in *pounds*, in which the others are expressed.]

By second hypothesis :—

$$x+\frac{14y}{5}=685.$$

Whence we shall find $x=125$.

Ex. 5. A bookseller asked £50 for a quantity of books, but consented to give a discount of 3*d.* in the shilling for the new ones, and 1*d.* in the shilling for the old. For the new books he received £4 3*s.* 4*d.* more than for the old; what price was given for the new and old respectively?

Let x be the price given for the new, y for the old, both in shillings.

Now, if x be the price after a discount of 3*d.* in the shilling has been deducted, the original price was $\frac{12}{9}x$ or $\frac{4}{3}x$.

So the price first asked for the old books was $\frac{12y}{11}$, since 11*d.* is paid for what 12*d.* was asked.

Hence, $\frac{4x}{3}+\frac{12y}{11}$ was the price asked, i.e. £50, which must be reduced also to shillings.

$$\therefore \frac{4x}{3}+\frac{12y}{11}=1000, \text{ or reducing,}$$

$$11x+9y=8250 \quad . \quad . \quad . \quad (1)$$

Also, by the second part of the question, $x-y=$ £4 3*s.* 4*d.* in shillings$=\frac{250}{3}$ . . . (2)

These equations give $x=$ £22 10*s.*, $y=$ £18 6*s.* 8*d.*

Examples (19).

(1) How can a debt of £13 4*s.* be paid by half-crowns and shillings, the number of half-crowns being greater by 7 than three times the number of shillings?

(2) A person could have given 6*d.* a piece to some poor persons if he had had 1*s.* more in his purse; he gave 5*d.* to each and had 8*d.* over. What was the number of poor persons, and how much money had he?

[In a case like this, though there seem to be two unknown elements, the question may be solved by the use of one equation only.]

(3) A painter works for 30 days and a mason for 18 days; their wages amounted to £11 17*s.* In 6 days the painter earned a shilling more than the mason did in 8. Find their daily wages.

(4) The value of a certain fraction when 3 is subtracted from numerator and denominator is $\frac{1}{4}$; but when 5 is similarly added its value is $\frac{1}{2}$. Find it.

(5) There are two numbers in the proportion 3 : 5; if 10 be added to the first and taken from the second, the proportion is reversed. Find the numbers.

(6) A sum of £5 was distributed among 45 old people, each man receiving half-a-crown, and each woman two shillings; how many were there of each sex?

(7) £24 are to be given to the best shots in a Volunteer Corps; some wished each long-range man to receive £2, and each short-range man £1 10*s.*; whilst others thought the best division would be to give £1 15*s.* to each of the former, and £1 13*s.* 9*d.* to each of the latter. How many prizemen were there?

(8) A grocer bought two kinds of tea, at 3*s.* and 4*s.* per lb., paying £50 in all. He sold the mixture at a profit of 1*s.* on every 2lbs., and gained £7 10*s.* How many lbs. did he buy of each kind?

(9) A number composed of two digits is equal to seven times its unit's figure; and if the digits be reversed, its value is increased by 18. Required the number.

(10) A courier travelled 240 miles in four days, diminishing his rate of travelling each day alike. If he travelled 136 miles in the first two days, find the number he travelled on the last day.

(11) Two taps running together would fill a cistern in 50 minutes, but in 35 minutes one is stopped up, and the cistern is full 27 minutes afterwards. In what time would each separately fill the cistern?

(12) A number consists of three digits, the right-hand one being 0; if the left-hand and middle digits be interchanged the number is diminished by 180; if the left-hand digit be halved, and the middle and right-hand digits be interchanged, the number is diminished by 454. Find the number?

(13) The diameter of half-a-crown is 1·3 inches, and of a florin 1·2 inches; 160 placed in contact in a straight line measure a pole; how many were there of each kind?

(14) Two lame men had a race which lasted 5 minutes; one had a start of 20 yards, but the other went half as fast again and won by 30 yards. Find the length of the course.

[In problems of this kind, if v be the space described in one unit of time, s the space described in t *such units*, $s = vt$, $\therefore v = \frac{s}{t}$, $t = \frac{s}{v}$ which forms are often required in constructing our equations.]

(15) A train, an hour after starting, meets with an accident which detains it an hour, after which it proceeds at three-fifths of its former rate, and arrives three hours behind time; but had the accident happened fifty miles farther on the line, it would have arrived an hour and twenty minutes sooner. Find the length of the line.

(16) A silversmith has two bars of silver weighing

21 and 17 ounces respectively; he makes an amalgam of 12 ounces of the first and 8 of the other, and finds its fineness 640 per mille; he makes another amalgam of the remaining portions, and its fineness is 650 per mille. What is the fineness of each bar?

[When an alloy is said to be 640 per mille fine, it is meant that 640 parts in 1000 are of pure metal.]

(17) A and B run a race to a post and back. A returning, meets B 90 yards from the post, and reaches the starting point 3 minutes before him. If he had then returned at the same pace, he would have met B at a distance from the starting place equal to $\frac{1}{6}$ the whole distance. Find the length of the course and the time of the race.

(18) The sum of the three digits which compose a number is 18; if the outside ones be interchanged, the number is increased by 198, but if the middle and right-hand digits be interchanged, the number is increased by 9. Find it.

(19) Two trains, 92 feet and 84 feet long respectively, are moving with uniform velocities on parallel rails; when they move in opposite directions they are observed to pass each other in a second and a half; but when they move in the same direction the faster train is observed to pass the other in six seconds. Find the rate at which each train moves.

(20) A postman starts from a station A with the mail bag, and delivers letters at a village B, and then at another village C, where he halts, having driven 22 miles. He returns for the second mail to A and delivers at B, where he rests, having gone 28 miles. Lastly he delivers at C, and returns for the night to A, having gone 24 miles. Find the distances of the villages from the station.

CHAPTER IX.

Quadratic Equations.

70. QUADRATIC EQUATIONS contain the square of the unknown quantity, but no higher power. There are two ways in which this may happen.

(1) The equation may contain the square of the unknown quantity, but not its simple power, and then it is called a *pure* quadratic.

(2) It may also contain the simple power of the unknown quantity, and is then called an *adfected* quadratic.

PURE QUADRATICS.

71. *Ex.* 1. $x^2-144=0$ $\therefore$ $x^2=144$, and taking the square root of each side, $x=12$. But since $(-12)^2=144$ also, $x=-12$ will also satisfy the equation; hence there are two roots, $+12$ and -12, which is thus expressed: $x=\pm 12$.

Ex. 2. $ax^2=b$. Every pure quadratic may clearly be reduced by simplification to this form, by collecting all the coefficients of x^2 to one side, and all the other terms to the other side. Dividing each side by a, $x^2=\frac{b}{a}$, and extracting the square root, $x=\pm\sqrt{\frac{b}{a}}$.

Ex. 3. $\frac{x-9}{x+3}-\frac{x-11}{x-3}=\frac{4}{9}-\frac{4x}{x^2-9}$.

$$x^2-12x+27-(x^2-8x-33)=\tfrac{4}{9}(x^2-9)-4x$$

or, $-4x+60=\tfrac{4}{9}(x^2-9)-4x.$

Striking out $-4x$, dividing each side by 4, and transposing :

$$\frac{x^2-9}{9}=15, \text{ whence } x^2=144 \text{ and } x=\pm 12.$$

ADFECTED QUADRATICS.

72. Every adfected quadratic may, by simplification, be reduced to the form $x^2+px+q=o$. For we may always collect the coefficients of x^2 together, and likewise the coefficients of x, and the terms which remain are known quantities. Then, dividing by the resulting coefficient of x^2, the equation is reduced to the above form. Thus we shall investigate the general rule by solving the equation $x^2+px+q=o$, in which p and q may be positive or negative.

73. To solve the equation $x^2+px+q=o$.

The equation may be written $x^2+px=-q$; now we know that $x^2+px+\frac{p^2}{4}$ is the square of $x+\frac{p}{2}$; hence in every case if we add to each side of an equation so reduced the square of half the coefficient of x, we make the left-hand side a perfect square. Thus :

$$x^2+px+\frac{p^2}{4}=\frac{p^2}{4}-q.$$

Take the square root of each side, and

$$x+\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}-q}$$

$$\therefore\ x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q};$$

and the two roots which satisfy the equation are

$$-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q} \text{ and } -\frac{p}{2}-\sqrt{\frac{p^2}{4}-q}.$$

Ex. 1. $2x^2-11x+12=0$

$$x^2-\frac{11x}{2}=-6.$$

Add $(\frac{11}{4})^2$ to each side, and

$$x^2-\tfrac{11}{2}x+(\tfrac{11}{4})^2=(\tfrac{11}{4})^2-6=\tfrac{25}{16}.$$

Take the square root of each side :

$$x-\tfrac{11}{4}=\pm\tfrac{5}{4};$$

whence $x=\frac{11}{4}+\frac{5}{4}=4$, or $x=\frac{11}{4}-\frac{5}{4}=\frac{3}{2}$.

Ex. 2. $\dfrac{5}{x-3}+\dfrac{3}{x-5}=2$. The learner may easily reduce this to $x^2-12x=-32$; adding 36 to each side (which is called completing the square) and extracting the root $x-6=\pm 2$, whence $x=8$ or 4. It may be seen by substituting in the given equation first 8, then 4, for x, that either root satisfies it.

The following examples of pure and easy adfected quadratics may now be worked out.

Examples (20).

(1) $11x^2+20=1879$.

(2) $\dfrac{5x^2}{17}=952-3x^2$.

(3) $\dfrac{8}{5x^2}-\dfrac{7}{3x^2}+18\frac{1}{3}=0$.

(4) $7x^2-3x=100$.

(5) $4(x^2+10x)+75=0$.

(6) $11x^2-8x=3$.

(7) $\frac{x^2}{2}-\frac{x}{3}=\frac{1}{32}$; $\frac{x^2}{8}-\frac{x}{3}=5\frac{1}{3}$.

(8) $x^2-25(2-x)=-116$.

(9) $\frac{12}{x}+6x=17$; $\frac{3x}{4}=1+\frac{1}{x}$.

(10) $x^2=\frac{x+1}{30}$; $\frac{x+1}{x}+\frac{x}{x+1}=2\frac{1}{6}$.

(11) $17x^2+19x-1848=0$.

(12) $\frac{7x}{3}+\frac{3-x}{2x}=20\frac{2}{3}$.

(13) $(x+7)(x-5)-\frac{2}{3}(5x-7)=23$.

(14) $2ax^2+(a-2)x=1$.

(15) $x^2-(m-n)x-mn=0$.

(16) $\frac{2x-3}{x-2}+\frac{2x-1}{1-x}=\frac{1}{6}$.

(17) $\frac{1}{x+a+b}=\frac{1}{x}+\frac{1}{a}+\frac{1}{b}$.

(18) $\frac{x^2}{a^2}-\frac{a^2}{b^2}=\frac{a^2}{b^2}-\frac{x}{b}$.

(19) $x+\frac{1}{x}+\frac{a}{b}+\frac{b}{a}=\frac{2(a^2+b^2)}{ab}$.

(20) $\frac{c}{a-c}\left(x+\frac{1}{x}\right)=1-\frac{a+c}{x(c-a)}+\frac{b}{a-c}\left(1+\frac{1}{x}\right)$.

74. We have seen that the roots of the equation $x^2+px+q=0$ (the general form to which every quadratic may be reduced) are $-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}$, and $-\frac{p}{2}-\sqrt{\frac{p^2}{4}-q}$. Hence a quadratic equation has two roots only.

Now if we add these roots together, the result is $-p$.

Hence, in any quadratic equation $x^2+px+q=o$, the sum of the roots is $-p$.

Again, if we multiply these roots together [the product of the algebraic sum of two numbers, $-\frac{p}{2}$ and $+\sqrt{\frac{p^2}{4}-q}$, by their difference $-\frac{p}{2}$ and $-\sqrt{\frac{p^2}{4}-q}$, is equal to the difference of their squares, $\left(-\frac{p}{2}\right)^2-\left(\sqrt{\frac{p^2}{4}-q}\right)^2$ or $\frac{p^2}{4}-\left(\frac{p^2}{4}-q\right)$ or q] : the result is q. Hence, in any quadratic equation x^2+px+q, the product of the roots is q.

75. It thus appears that if α, β be the roots of $x^2+px+q=o$; $\alpha+\beta=-p$, $\alpha\beta=q$; hence, substituting, the equation may be written $x^2-(\alpha+\beta)x+\alpha\beta=o$

$$\text{or } (x-\alpha)(x-\beta)=o.$$

Hence we may form an equation with any given roots.

76. Returning again to the roots of the equation $x^2+px+q=o$, viz. : $-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$, we observe—

(1) If $\frac{p^2}{4}=q$ the second term of each root vanishes, and the two roots are equal in value. This also appears at once from the equation, for writing $\frac{p^2}{4}$ instead of q, it becomes $x^2+px+\frac{p^2}{4}=o$,

or $\left(x+\frac{p}{2}\right)^2=o$, and $x=-\frac{p}{2}$.

(2) If $\frac{p^2}{4}$ be greater than q, $\sqrt{\frac{p^2}{4}-q}$ will be positive, and the two roots will be real and different in value.

(3) If $\frac{p^2}{4}$ be less than q, $\sqrt{\frac{p^2}{4}-q}$ will be negative, and the roots are said to be impossible, since we cannot extract the square root of a negative quantity.

To sum up these results, if an equation be reduced to the form $x^2+px+q=o$, its roots will be real and equal if $p^2=4q$, real and unequal if p^2 be greater than $4q$, impossible if p^2 be less than $4q$.

Of course, though the roots may be real it does not follow that we can exactly find them : we cannot exactly obtain $\sqrt{2}$, $\sqrt{3}$, &c., though we may approximate to them as nearly as may be required. Such quantities are termed surds.

77. The following is another proof of the important proposition, 'the sum of the roots of the equation $x^2+px+q=o$ is $-p$, and their product is q.'

Let the roots be α, β. Then, since (by the definition of a root) they satisfy the equation :

$$\alpha^2+p\alpha+q=o$$
$$\beta^2+p\beta+q=o$$

Subtracting the latter from the former :

$$\alpha^2-\beta^2+p(\alpha-\beta)=o$$

Divide by $\alpha-\beta$, and

$$\alpha+\beta+p=o \quad \therefore \quad \alpha+\beta=-p.$$

Now substitute $-(\alpha+\beta)$ for p in the first equation, and $\alpha^2-\alpha(\alpha+\beta)+q=o$, whence $q=\alpha\beta$.

78. A great variety of equations not exactly cast in the form of a quadratic may be reduced to that form by analytical ingenuity and the application of artifices of different kinds. Some of the following examples fall under classes which often occur : it is obvious that there is no general rule for solving difficult equations, but, on

the other hand, we may promise the learner that by practice he will find few cases where he cannot read the riddle of the equation in a few minutes.

79. In many kinds of equations it is necessary to square both sides in order to reduce the equation to the normal form. It is important to observe that, in such cases, some of the values we ultimately find for the unknown quantity may belong to another equation, from which the same expression would be deduced, after squaring both sides, as we obtained from our proposed equation. This is an important remark which will be made clearer by an example.

Let the equation be $ax+\sqrt{bx+c}=d$; then $ax-d=-\sqrt{bx+c}$, and squaring both sides to get rid of the root :--

$(ax-d)^2=bx+c$ which may be solved in the usual way.

But suppose the equation had been

$$ax-\sqrt{bx+c}=d.$$

$ax-d=\sqrt{bx+c}$, and squaring as before :

$(ax-d)^2=bx+c$; precisely the same form as before. In such cases we must ascertain by trial which of the values obtained for x belong to the original equation.

Ex. 2. $3x-2\sqrt{2x+1}=6$

$$3x-6=2\sqrt{2x+1}$$

$$9x^2-36x+36=8x+4$$

$$9x^2-44x=-32$$

This solved gives $x=4$ or $\frac{8}{9}$.

The value $x=4$ belongs to our equation ; $x=\frac{8}{9}$ is a root of $3x+2\sqrt{2x+1}=6$, which after squaring would give the same quadratic $9x^2-44x=-32$.

Examples (21).

(1) Form the equation whose roots are 6 and $-\frac{3}{2}$.

[See Section 74.]

(2) Form the equation whose roots are 0 and 7.

(3) In a quadratic equation the coefficient of x and also of x^2 is 1; if one root be -5, what is the other root?

[Section 73.]

(4) If one root of the quadratic $x^2+px+b^2-a^2=0$ be $a+b$, find p.

(5) $\dfrac{x+2}{x-1}-\dfrac{4-x}{2x}=2\frac{1}{3}$; find x.

(6) $\dfrac{x+4}{x-3}-\dfrac{2x-3}{x+4}=7\frac{3}{8}$.

(7) $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}=\dfrac{x}{5}-\dfrac{38}{15}$.

(8) $\dfrac{x^2+ax+b}{x^2+bx+c}=\dfrac{a}{b}$.

(9) $x^2+9x+20=0$.

[Solve this by at once resolving into factors.]

(10) $\cdot07\,x^2-\cdot625x+1\cdot245=0$.

(11) $(x-2a+3b)^2-3a\,(x-2a+3b)-4a^2=0$.

[Any such equation may at once be solved by writing y for the expression which appears in its first and second power: then having solved $y^2-3ay-4a^2=0$, substitute for y.

Some equations, though not proposed in this form, may be reduced to it by a little ingenuity.]

(12) $x^4-25x^2+144=0$.

[An *equation* containing x^4 and x^2 but not x nor x^3 may be *solved as a quadratic*; completing the square in the usual *manner, we shall* obtain two values for x^2, and taking the

square root of each value, four roots of the equation will be found.]

(13) $9(9x^2-2)(x^2-3)+46=0$.

(14) $x^3+8=2x^2+11x+14$.

[If an equation of the third degree can be thrown into the form $(x-a)(x^2+px+q)=0$ by the detection of a factor $x-a$, then it is satisfied either by putting $x-a=0$ or $x^2+px+q=0$. The former gives $x=a$, one solution; from the latter two other roots may be obtained. In the above example $x+2$ is a factor common to both sides.]

(15) $x(x-3)^2+2x=6$.

(16) $x^2+5x+\sqrt{x^2+5x-11}=41$.

[When an equation contains terms involving x^2 and x, and the square root of the *same* terms with some known quantity, it may be solved as a quadratic. In the example, we may write the equation, $x^2+5x-11+\sqrt{x^2+5x-11}=30$. Now put y^2 for $x^2+5x-11$, and the equation will be written $y^2+y=30$, which solved gives $y=5$ or $y=-6$. First take $y=\sqrt{x^2+5x-11}=5$ $\therefore$ (squaring) $x^2+5x-11=25$, whence we obtain two roots; and secondly from $\sqrt{x^2+5x-11}=-6$ we shall get two other roots.

Observe that when we put y^2 for $x^2+5x-11$, then $y=\pm\sqrt{x^2+5x-11}$. Hence, just as in section 78, all our values do not apply to the given equation, since the same process would be carried out if the given equation were $x^2+5x-\sqrt{x^2+5x-11}=41$. The reader should bear in mind the observations made in section 78 when working the following questions.]

(17) $x+\sqrt{2x+3}=16$.

(18) $2x+17=9\sqrt{2x-1}$.

(19) $9x+2x\sqrt{9x+4}=15x^2-4$.

[Transpose -4, add x^2 to each side, take the square root of each side.]

(20) $x+\sqrt{5x+10}=8$.

(21) $2x-\sqrt{x^2-3x-3}=9$.

(22) $x^4 - 13x^2 + 36 = 0.$

(23) $x^2(x+3a) = 4a^3.$

(24) $\dfrac{a - \sqrt{2ax - x^2}}{a + \sqrt{2ax - x^2}} = \dfrac{x}{-x}.$

[We have shown in fractions if $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, &c. Equations like that just given may easily be solved by applying this principle. Thus taking the numerator from the denominator for a new numerator, and adding them for a new denominator, on each side, we have :—

$$\frac{2\sqrt{ax - x^2}}{2a} = \frac{a-2x}{a}, \text{ or reducing :—}$$

$$\sqrt{2ax - x^2} = a - 2x.$$

Square both sides, and

$$2ax - x^2 = a^2 - 4ax + 4x^2$$
$$\text{or } 5x^2 - 6ax + a^2 = o,$$

$\therefore$ by inspection $(5x-a)(x-a) = o$, and

$x = \frac{a}{5}$ or $x = a$. Which value satisfies the equation ?]

(25) $\dfrac{x + \sqrt{12a^2 - x^2}}{x - \sqrt{12a^2 - x^2}} = \dfrac{a+1}{a-1}.$

(26) $\dfrac{\sqrt{36x+1} + \sqrt{36x}}{\sqrt{36x+1} - \sqrt{36x}} = 9.$

(27) $\dfrac{\sqrt{ax} - b}{\sqrt{ax} + 3b} = \dfrac{ax - 2b}{\sqrt{ax} + 5b}.$

(28) Find the value of the sum of the reciprocals of the roots of the equation $x^2 + px + q = o.$

[If α be a root, $\frac{1}{\alpha}$ is its *reciprocal.* We know that $-p = \alpha + \beta(1)$; $q = \alpha\beta(2)$;—divide (1) by (2).]

PROBLEMS WHICH PRODUCE QUADRATIC EQUATIONS.

We have already discussed a variety of questions whose conditions can be expressed in an algebraical sentence, whence, by applying algebraical rules, the unknown element or elements may be determined. The algebraical sentences involving the conditions of the problem have hitherto contained the unknown element in its first power only, but numerous problems give rise also to quadratic equations. Indeed, it is evident that every quadratic equation is the expression of some problem or other; thus the equation $x\ (x+1)=702$ is the algebraical expression of this problem: 'The product of two consecutive numbers is 702 : find them.' There is however one important remark which applies to problems producing equations of a degree higher than the first; it is this: Every quadratic equation has two roots, yet a problem giving rise to a quadratic may obviously admit of but one solution. What does the other root, which does not seem to satisfy the conditions of the problem, signify? The fact is, our algebraical sentence is often more general than ordinary language, and may apply to other conditions than those of the problem which gives rise to it, and may answer other suppositions.

Thus, if we take the problem just referred to—'The product of two consecutive numbers is 702 : find them,' the solution of the equation $x\ (x+1)=702$ is $x=26$ or -27. Thus the positive root tells us that one of the numbers is 26. But also if negative numbers were not excluded $x=-27$, and $x+1=-26$, and we see that $-26\times-27=702$.

Ex. 2. Again, take the following problem, the type of a numerous class of questions. A certain number of

things were bought for £96; had 8 more been received for the same money, each thing would have cost £1 less. Required the number and the price of each.

Let x be the number of things, then the price of each is $\frac{96}{x}$, and the price of each if $x+8$ had been received is $\frac{96}{x+8}$; ∴ by conditions of question

$$\frac{96}{x}-1=\frac{96}{x+8}, \text{ whence}$$

$$x^2+8x=768, \text{ which gives}$$

$$x=24 \text{ or } -32.$$

Thus the answer is 24, the other root referring to another problem. Remembering what was said about the significance of negative quantities in one of the early chapters of this book, it will not be difficult to draw up the statement of this problem; it is this: A certain number of things were *sold* for £96; had 8 *less* been sold for the same money, each thing would have cost £1 *more*. Required the number and the price of each.

The reader may work this out, comparing each step with the similar step in the former case.

Ex. 3. To divide a given line into two parts, so that the square on the one part may be double the square on the other.

Let the length of the line be a, and let one of the parts be x; then $(a-x)^2=2x^2$, which gives $x=a\sqrt{2}-a$ or $-(a\sqrt{2}+a)$. The first (positive) root indicates the solution.

Suppose however the problem ran thus. To produce a given line so that the square on the whole line thus produced may be double the square on the part produced.

Let the length of the line be a and let the part produced be x; then $(a+x)^2=2x^2$, which gives
$x=-(a\sqrt{2}-a)$ or $a\sqrt{2}+a$; the latter being the solution. Compare the two statements and the roots of the two equations.

See part I. of this course, p. 113; and chapter ii., section 20, of part II.

Examples (22).

(1) The product of two consecutive numbers is 650: find them.

(2) If $\frac{1}{4}$ be added to the product of any two consecutive numbers, the result is a perfect square.

(3) The difference between the sum of the squares of two consecutive numbers and their sum is 2 more than 14 times the larger number: find them.

(4) Find two numbers such that each added to nine times its reciprocal is equal to ten.

(5) A grazier bought sheep for £120, out of which he kept 15, and gained 4 shillings a head on the remainder by selling them for £108. How many did he buy?

(6) A woman spent £5 in buying fowls, out of which she reserved 20, and gained 4*d.* a head on the remainder by selling them for £4. How many did she buy?

(7) The difference of two numbers is 4, and the sum of their reciprocals is $\frac{5}{24}$: find the numbers.

(8) A man by walking at the rate of $\frac{1}{3}$ of a mile per hour above his ordinary rate of walking gains $\frac{3}{4}$ of an hour in 39 miles: at what rate does he walk?

(9) Find two numbers in the proportion of 9 : 7, so that the square of their sum shall be equal to the cube of their difference.

(10) A merchant sells two casks of wine for £76 5*s.*;

one holds 5 gallons more than the other, and the price of each wine is in shillings the number of gallons in the cask which contains it. How many gallons are there in each cask?

(11) A vessel can be filled by two pipes; by one of them alone the vessel would be filled two hours sooner than by the other; and by both together it would be filled in $112\frac{1}{2}$ minutes. In what time would each alone fill the vessel?

(12) A captain paid £5616 for horses for his troop, and got 4 less than in the previous year for the same money, when he did not pay so much by £2 for each. How many did he buy?

(13) The difference of two numbers is 8, and the sum of their squares is 5 times the square of the smaller one. Find them.

(14) What do eggs cost per dozen, when two less in a shilling's worth raises the price a penny per dozen?

(15) The cost of an entertainment was £6, which was to have been divided equally among the party, but four of them leave without paying, and the rest have each to pay half-a-crown extra in consequence. Of how many persons did the party consist?

(16) A man bought some oxen for £240, and after losing three, sold the remainder for £8 a head more than they cost him, thus gaining £59 by his bargain. Required the number he purchased.

(17) A boy bought marbles at so much per dozen, but if he could have afforded to spend half-a-crown, he would have got them at a halfpenny less per score, and he would have received 60 more than half-a-crown's worth at the price he paid. What did he pay per dozen?

(18) A man drew some wine from a full vessel which

held 81 gallons, and filled it up with water. He then drew from the mixture as much as he drew before of pure wine: it was found that 64 gallons of pure wine remained. How much did he draw each time?

(19) A and B together can do a piece of work in $14\frac{2}{5}$ days, and A working alone can do it in 12 days less than B working alone. Find the time in which A can do it alone.

(20) A man leases some land for £150 a year; he retains 15 acres for his own use, and lets the rest for building at 10 shillings per acre more than he gave for it. By so doing he just paid his rent. How much land did he let?

(21) A party of friends incurred a hotel charge of £19 16*s.*, which was equally divided amongst all but two, each share being thus 2*s.* 9*d.* more than if the charge had been equally divided amongst all. What number did the party consist of?

(22) A father and his son can dig a garden in $3\frac{3}{5}$ days: were they to work singly, the son would take 3 days more than the father. In what time would each dig it, working alone?

(23) A cricket club entertained an eleven of another club at luncheon, the share of each member being $5\frac{1}{2}$*d.* more than it would have been if all had paid, and the cost was £4 5*s.* What was the number of members?

(24) By selling a horse for £119 I gain as much per cent. as the horse cost. What did it cost?

(25) A man bought some sheep at Rotterdam for £330, and paid 6*s.* 8*d.* per head for their transport to London, where, after 17 had died, he sold the rest at £3 per head, thus gaining on the whole £8 more than 5 sheep had originally cost. How many did he buy?

(26) A and B set off at the same time from two towns

90 miles apart, each proceeding to the other town. A, walking two miles an hour faster than B, accomplished the journey 7 hours and 30 minutes before B. Find B's rate of travelling, and the point where they met.

(27) Two vessels, one of which sails faster than the other by 2 miles an hour, start together on voyages of 1152 and 720 miles respectively; the slower vessel reaches its destination one day before the other. How many miles per hour did the faster vessel sail?

CHAPTER X.

Examination Papers.

THE following fifty-two examination papers are given in two series, twenty-six in each. The first twenty-six papers contain questions to the end of simple equations of one unknown quantity, with the exception of cube root. The second series contain questions on the whole of this book, but with especial reference to the last two chapters. Every kind of difficulty found by beginners is represented.

For pupil-teachers, the following plan of work is recommended. Those in the fourth* year may learn the bookwork and work the examples in the first six chapters during the first six months of the year; and on each Saturday during the remainder of the year, they may work out one of the papers in the first series, taking them in order. When the master looks over the work each week, it will be a great advantage if he fully explains the difficulties, and sees that the pupil-teacher revises the bookwork, where his answers prove him to be deficient. Similarly pupil-teachers in the fifth* year may revise the first six chapters thoroughly and learn the two others during half the year, and work out one of the second series of papers each week of the remaining half-year. All will at least be able to do this, and if they do it, they will complete their course in algebra with credit.

* See note to the Preface.

FIRST SERIES.

A.

1. Add together $2a^2+ab-3b^2$, $a^2-2ab+4b^2$, $-4a^2+5ab-6b^2$; and subtract $2a-4b+5c-6d$ from $3a+6b-8c-d$.

2. Divide $a^4-9a^2-6ab-b^2$ by a^2+3a+b.

3. Reduce the following expressions to their simplest forms :—

$$(1)\ \frac{a^3-3a^2b+3ab^2-b^3}{a^4-2a^2b^2+b^4}.$$

$$(2)\ \frac{1}{x+3y}+\frac{6y}{x^2-9y^2}-\frac{1}{3y-x}.$$

4. State and explain the rule for transferring a quantity from one side of an equation to the other side.

Solve the equations :—

$$(1)\ x-1-2(2x-9)=\frac{2x}{5}.$$

$$(2)\ \frac{x-\frac{1}{a}}{c}+\frac{x-\frac{1}{b}}{a}+\frac{x-\frac{1}{c}}{b}=0.$$

B.

1. Find the sum and the difference of the two expressions : $a-9\{6b-a+3(a-4b)\}$ and $b-9\{6a-b+3(b-4a)\}$.

2. Multiply x^2+x-2 by x^2+x-6; and divide the result by x^2+5x+6.

3. Prove the rule for adding fractions together.

Simplify $\frac{1}{x-1}-\frac{1}{2x+2}-\frac{x+3}{2x^2-2}$.

4. Solve the equations :—

$$(1)\ 2x+3=16-(2x-3).$$

$$(2)\ x+\frac{1}{x}=\frac{(x+1)(x+2)}{x}.$$

C.

1. Find the difference between

$$\sqrt[3]{a^2+2b^2+3c^2+1+2bc}$$

and $\sqrt{(a+c)^3-(2b)^3}$, when $a=1$, $b=3$, $c=5$.

2. Divide $3a^3+4a^2xy-6ax^2y^2-4x^3y^3$ by $2xy+a$.

3. Prove the rule for finding the G. C. M. of two algebraical expressions, and find it for—

$$a^2+ab-12b^2 \text{ and } a^2-5ab+6b^2.$$

4. Solve the equations :—

$$(1)\ -7x-22=4x-(21x-8).$$

$$(2)\ \frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36.$$

D.

1. Find the value of the expression

$$ab^2-2bc^2+3cd(b^2a-3a^2c)+4ac(bd-ac),$$

(1) when $a=3$, $b=-3$, $c=-2$, $d=2$;

(2) when $2a=b=-c=2d=\frac{1}{3}$.

2. On what principle is the rule for division in algebra based?

Divide $x^5+2x^4-10x^3-19x^2+11x+15$ by x^3-7x-5.

3. Reduce $\dfrac{x^2+2x-3}{x^2+5x+6}$ and $\dfrac{x^3-2x^2}{x^2-4x+4}$ to lowest terms.

4. Divide $\frac{276}{277}$ into two parts differing by $\frac{7}{9}$.

E.

1. From $3a-b-9[2b-3\{a-2(b-a)\}]$ take $18a-91b-3\{6a-2(3b-a)\}$.

2. Prove the rule of signs in multiplication. Multiply $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$ by $2b+a$.

3. Find the G. C. M. of $4x^3+5a^2x-21a^3$ and $2x^3+5ax^2-30a^2x+27a^3$.

4. Solve the equations :—

   (1) $3-(-7x+6)=2-(-3x)$.

   (2) $\dfrac{x+3}{4}+\dfrac{20-x}{5}+2\frac{2}{3}=x+2-\dfrac{x-7}{3}$.

F.

1. Explain *term*, *power*, *like quantities*, *literal coefficient*, with examples ; and prove that if a, m, n be any positive numbers, $a^m \times a^n = a^{m+n}$.

2. Multiply $3x^3-4ax^2+7a^2x-9a^3$ by $4x-5a$, and divide a^6+a^4-2 by a^4+2a^2+2.

3. Reduce to lowest terms

$$\frac{3a^4-a^2b^2-2b^4}{10a^4+15a^3b-10a^2b^2-15ab^3}.$$

4. Solve the equations :—

   (1) $bx+2x-a=3x+2c$.

   (2) $\dfrac{11x-13}{25}+\dfrac{19x+3}{7}-\dfrac{5x-25\frac{1}{3}}{4}=28\frac{1}{7}-\dfrac{17x+4}{21}$.

G.

1. Divide 1 by $2a-x$ to four terms, and multiply x^2-px+q by $qx-p-1$.

2. Prove the rule for multiplying one fraction by another, and multiply—

$$\frac{3xy(x^2-y^2)}{x^2-3xy+2y^2} \text{ by } \frac{x-2y}{x^2(x+y)^2}.$$

3. Find the square root of $x^4-2x^3+\frac{3}{2}x^2-\frac{x}{2}+\frac{1}{16}$.

4. Solve the equations :—

$$(1)\ \frac{x+1}{2}+\frac{x+2}{3}=14+\frac{5-x}{4}.$$

$$(2)\ \frac{x-7}{11}-\frac{3x-5}{7}+\frac{125}{77}=2x-17.$$

H.

1. Explain how we may consider a negative quantity to indicate a debt or loss.

Take $(a+b)x+(b+c)y+(c+d)z$ from $(a+c)x+2cy+(b+d)z$, and show that the result may be written $(b-c)(z-x-y)$.

2. Find the L.C.M. of x^3+x^2+x+1 and x^3-x^2+x-1.

3. Add together:—

$$\frac{a^2+x^2}{a^3+x^3},\ \frac{2(a+x)}{a^2-ax+x^2},\ \frac{a}{a^2-x^2}.$$

4. Solve the equations :—

$$(1)\ \frac{x}{4}-\frac{x-4}{6}=\frac{4}{3}+\frac{24-x}{12}.$$

$$(2)\ \frac{4x-34}{17}-\frac{258-5x}{3}=\frac{69-x}{2}.$$

I.

1. Divide $2a^2+3b^2+4c^2+5ab-6ac-7bc$ by $a+b-c$; also $x^3-8y^3+125z^3+30xyz$ by $x-2y+5z$.

2. Prove the rule for the addition of fractions.

Add together :—

$$\frac{x-a}{x^2-ax+a^2},\ \frac{x+a}{x^2+ax+a^2},\ \text{and}\ \frac{2(a^3-x^3)}{x^4+a^2x^2+a^4}.$$

3. Solve the equations :—

(1) $x=3x-\frac{4-x}{2}+\frac{1}{3}$.

(2) $\frac{1}{6}(x+3)-\frac{1}{7}(11-x)=\frac{2}{5}(x-4)-\frac{1}{21}(x-3)$.

4. A farmer sold to one person six more than one fourth of his sheep, to another one fifth of the remainder, and to a third two more than one sixth of those now left. If he has still eighteen unsold, how many had he at first?

J.

1. Simplify $5x-2y-[3x-2\{y-3x-4(x-y)\}]$ and multiply the result by $8y+12x$.

2. Show that in finding the G.C.M. of two algebraical expressions, the result is not affected if at any step we multiply the dividend by a simple factor, or if we strike out any simple factor.

Find the G.C.M. of $x^3-3x^2+7x-21$ and $2x^4+19x^2+35$; and of $20x^4+x^2-1$ and $5x^4+5x^3+4x^2-x-1$.

3. Find the square root of :—

$$36x^4-132x^3y+277x^2y^2-286xy^3+169y^4.$$

4. Solve the equations :—

(1) $\frac{x+6}{4}-\frac{16-3x}{12}=\frac{25}{6}$.

(2) $\frac{4x+3}{2x+1}-\frac{3}{3x+2}=\frac{4x-9}{2x+1}$.

K.

1. Add $a(b+c)-2bc$, $b(c+a)-2ac$, $c(a+b)-2ab$; and from $(a-b)x-(b-c)y$ take $(a+b)x+(b+c)y$. Show that your result in the latter case is divisible by $x+y$.

2. Prove the rule for reducing a fraction to its lowest terms.

Reduce $\frac{85(20a^4+a^2-1)}{102(25a^4+5a^3-a-1)}$.

3. Find the square root of $x^2+\frac{1}{x^2}+2\left(x-\frac{1}{x}\right)-1$.

4. Solve the equations :—

(1) $\frac{x}{2}-\frac{5x+4}{3}=\frac{4x-9}{3}$.

(2) $\frac{ax-b^2}{a-b}-\frac{bx-a^2}{a+b}=\frac{ab}{b-a}$

L.

1. Remove the brackets from $a^2-[(b-c)^2-\{c^2-(a-b)^2\}]$ and find the numerical value of $x^3+(p-3)x^2+(q-3p)x-3q$ when $x=3$.

2. If c be a common measure of a and b it will also measure $ma+nb$ and $ma-nb$.

Find the G.C.M. of $2x^4-3x^3+2x^2-2x-3$ and $3x^4-4x^3-x-2$.

3. Add $\frac{a-b}{(a+c)(b+c)}+\frac{b-c}{(b+a)(c+a)}+\frac{c-a}{(c+b)(a+b)}$.

4. What is meant by an *identity*? Give examples.

Solve the equations :—

(1) $x-\frac{x-7}{3}+\frac{3x-1}{5}-\frac{2x}{7}=9.$

(2) $9-\frac{2x-3}{2x-2}=\frac{5x+3}{x-1}.$

M.

1. Divide $x^5-2x^4-4x^3+19x^2-31x+15$ by x^3-7x+5; also find the quotient to four terms and the remainder of $a\div(a+b)$.

2. Show how the square root of an algebraical expression may be found.

Find the square root of $x^4-6x^3+13x^2-12x+4$.

3. Simplify the expressions :—

(1) $\left(\frac{x+y}{x-y}\right)^2-\left(\frac{x-y}{x+y}\right)^2$;

(2) $\frac{x+y}{x}-\frac{x^2+y^2}{x(y-x)}-\frac{2y}{x+y}$,

and divide the first by the second.

4. Solve the equation :—

$$\frac{5x-1}{2}-\frac{7x-2}{10}=6\tfrac{3}{5}-\frac{x}{2}.$$

N.

1. Divide $x^3-63-40x$ by $x-7$, and $1-6x^5+5x^6$ by $1-2x+x^2$.

2. Find the G. C. M. of $x^3-12x-16$ and x^3-2x^2-32; also of $x^3-8x^2+11x+20$ and $2x^4-7x^3-4x^2+x-4$; and from your two results, write down the G. C. M. of the four expressions.

3. Solve the equations :—

$$(1)\quad \dot{.}25x - 1 = .22\dot{5}x - \frac{1}{2}.$$

$$(2)\quad \frac{x-3}{x-2} + \frac{x-4}{x-3} = 2.$$

4. Divide £1 among four children, so that the eldest may receive 1*s.* more than the second, the second 1*s.* more than the third, and the third 1*s.* more than the youngest.

O.

1. Resolve into factors :—

(1) $36x^2 - 49y^2$,
(2) $16x^2 - 40xy + 25y^2$,
(3) $x^2 - 12xy + 35y^2$,
(4) $2x^3 - x^2y - 6xy^2 + 3y^3$.

2. Find the G. C. M. of $27a^5b^2 - 18a^4b^2 - 9a^3b^2$ and $36a^6b^2 - 18a^5b^2 - 27a^4b^2 + 9a^3b^2$.

3. Prove the rules for multiplying and for dividing a fraction by an integer.

Multiply together $\frac{1-a^2}{b+b^2}$, $\frac{1-b^2}{a+a^2}$, and $b + \frac{ab}{1-a}$.

4. Solve the equations :—

$$(1)\quad \frac{3x+4}{14} - \frac{2x-7}{21} + 2\tfrac{3}{4} = \frac{x-4}{4}.$$

$$(2)\quad \frac{x-1}{x+3} - \frac{2x-1}{3x+1} = \frac{1}{3}.$$

P.

1. Multiply $y^6 - b^2y^4 + b^4y^2 - b^6$ by $y^6 + b^2y^4 + b^4y^2 + b^6$, and divide the product by $y^4 - b^4$.

2. Find the G. C. M. of :—

(1) a^3x^2y, x^3y^3, ax^3y^2, and b^3cxy;

(2) $x^4-x^3-7x^2+13x-6$ and
$x^5-11x^3+6x^2+28x-24$.

3. Explain the terms *involution*, *evolution*, and find the square root of :—

(1) $x^4-4x^3+10x^2-12x+9$.

(2) $\frac{x^4}{a^4}+\frac{a^4}{x^4}+\frac{x^2}{a^2}+\frac{a^2}{x^2}+2\left(\frac{x^3}{a^3}-\frac{a^3}{x^3}\right)-2\left(\frac{x}{a}-\frac{a}{x}\right)$.

4. Solve the equations :

(1) $\frac{x-3}{4}-\frac{x-1}{9}=\frac{x-5}{6}$.

(2) $\frac{3x+4}{2x+1}-\frac{1}{2}=\frac{x+19}{x+12}$.

Q.

1. Prove the rule for subtracting one algebraical expression from another, and take

$(p-q)^2x^3-3pqx^2-(p+q)^2x-q^3$
from $(p^2+q^2)(x^3-x)-7pqx^2-q^3$.

Show that your result is equal to $2pqx(x-1)^2$.

2. Find the G. C. M. of $x^2+11x+30$ and
$9x^3+53x^2-9x-18$.

3. Solve the equations :—

(1) $x-\frac{x-2}{3}=5\frac{3}{4}-\frac{10+x}{5}+\frac{x}{4}$.

(2) $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$.

4. A man finds that by transferring his money from the 3 per cents. at $86\frac{1}{4}$ to the $4\frac{1}{2}$ per cents. at $103\frac{1}{2}$ he will increase his income by £30 ; what money had he in the 3 per cents.?

R.

1. Reduce to its simplest form—

$$\{(1+x)-(1+2x)\}+\{(1-x)+(1-2x)\}$$
$$-\{(1-x)-(1-2x)\}$$; and find the value of
$$(a-x)^2-(3b-x^2)+\sqrt{(a-x)(b+y)}$$
when $a=16$, $b=10$, $x=5$, $y=1$.

2. Find the G. C. M. of $3x^4-x^2y^2-2y^4$ and $10x^4+15x^3y-10x^2y^2-15xy^3$.

3. Prove the rule for dividing one fraction by another, and simplify:

$$\left\{\frac{2a}{(x-2a)^2}-\frac{x-a}{x^2-5ax+6a^2}+\frac{2}{x-3a}\right\}\div(x-2a)^2.$$

4. A man paid 10 guineas for 20 yards of cloth, for part of which he gave 11*s.* 6*d.* a yard, and for the rest 7*s.* 6*d.* a yard. How much did he buy of each sort?

S.

1. From $7x\{2y-4(x-3y)\}$ take $3x(y-x)-5y(2x-y)$; also multiply $2a^2-3ab+b^2$ by b^2+2a^2+3ab.

2. Simplify $\dfrac{a^3+3a^2x+3ax^2+x^3}{x^3-y^3}\div\dfrac{(a+x)^2}{x^2+xy+y^2}$.

3. Solve the equations :—

(1) $\dfrac{2x-10}{3}-15=\dfrac{3x-40}{11}-\dfrac{57-x}{5}$.

(2) $\dfrac{x+1}{x-1}-\dfrac{x-9}{x-7}=\dfrac{x}{x-2}-\dfrac{8-x}{6-x}$.

4. A mixture is made of a gallons at p shillings per gallon, b gallons at q shillings, and c gallons at r shillings. What will be its value per gallon?

T.

1. Divide a^6+b^6 by a^2+b^2, and find the square of the quotient.

2. Find the G. C. M. of $x^7-3x^6+x^5-4x^2+12x-4$ and $2x^4-6x^3+3x^2-3x+1$.

3. Reduce to lowest terms :—

$$(1)\ \frac{ac+bd+ad+bc}{af+2bx+2ax+bf}.$$

$$(2)\ \frac{2y^3+y^2-8y+5}{7y^2-12y+5}.$$

4. In a mile race between a bicycle and a tricycle, their rates were proportional to 11 and 8. The tricycle had 320 yards start, but was beaten by half a minute. Find the rates of each.

U.

1. Multiply $\frac{x^2}{2}-\frac{xy}{3}+y^2$ by $\frac{x^2}{2}+\frac{xy}{3}+y^2$,

and divide $1+4a^4b^4$ by $2a^2b^2-2ab+1$.

2. Show how to find the L. C. M. of two or more expressions, and find it in the case of x^3-2x^2-3x+6, x^3-2x^2-4x+8, x^4-7x^2+12.

3. Simplify :—

$$(1)\ \frac{x}{1-x}-\frac{x^2}{(1-x)^2}-\frac{2x^3}{(1-x)^3}.$$

$$(2)\ \frac{\frac{a}{b}+\frac{b}{a+b}}{a^2-ab+b^2-\frac{2b^3}{a+b}}.$$

4. A person distributed p shillings among n persons, giving 9*d.* to some and 15*d.* to the rest, respectively. How many were there of each?

Discuss the case (1) when $5n=4p+2$; (2) when $n=\cdot 8p$.

V.

1. Prove from the definition of the algebraical symbols that $(a+b)(c-d)=ac-ad+bc-bd$, a, b, c, d, being positive numbers.

2. In finding the G. C. M. of two given algebraical expressions, can we reject either altogether or temporarily a factor occurring in *both* expressions? Give a reason for your answer.

Find the G. C. M. of $(a+2c)(a-c)-b(4a-3b-c)$, and $(a+2b)(a-b)-c(4a-b-3c)$.

3. Solve the equations:

$$(1)\quad \frac{2x-\frac{5}{2}}{2}-\frac{2x-\frac{3}{2}}{5}=\frac{4x}{13}.$$

$$(2)\quad -2+\frac{6x}{4+2x}=\frac{ax}{ax+a}.$$

4. How many pounds of sugar at $4\frac{1}{8}d.$ per lb. must a grocer mix with 110 lbs. at $3\frac{3}{4}d.$ in order that he may gain 20 per cent. by selling the whole at 5*s.* 6*d.* per stone?

W.

1. Divide $x^3-2ax^2+(a^2+ab-b^2)x-a^2b+ab^2$ by $x-a+b$.

2. Prove that when $\frac{x}{a}$ is a small fraction, $a+\frac{x}{2a}$ is nearly the square root of a^2+x. Find an approximate value for $\sqrt{145}$.

3. Simplify $\left(\frac{1}{1+x}+\frac{x}{1-x}\right)\div\left(\frac{1}{1-x}-\frac{x}{1+x}\right)$.

4. Suppose a creature, capable of seeing objects at any distance, to fly from the earth into space: at what rate must he move, in order that in two years hence he may see the earth as it was 5873 years ago; the velocity of light (with which the sight of distant objects is conveyed to us) being 192,000 miles per second.

X.

1. Divide $mpx^3 + (mq - np)x^2 - (mr + nq)x + nr$ by $mx - n$.

2. Prove that the difference between the cube of the sum of any two numbers and the sum of their cubes s divisible by three times their product.

3. Solve the equations :—

(1) $\cdot 15x + \cdot\dot{2} - \cdot 875x + \cdot 375 = \cdot 0625x - 1$.

(2) $\frac{4x - 21}{7} + 7\frac{5}{6} + \frac{7x - 28}{3} = x + 3\frac{3}{4} - \frac{9 - 7x}{8} + \frac{1}{12}$.

4. A grocer bought 15 stones of sugar at 7*s.* 4*d.* per stone. The sugar having been damaged, he sold a portion of it at $5\frac{1}{2}$*d.* per lb., and the rest at 5*d.*, and on the whole transaction lost $16\frac{2}{3}$ per cent. How much did he sell at each price?

Y.

1. Multiply $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ by $\frac{1}{a} - \frac{1}{b} + \frac{1}{c}$. What does the result become when $a = b = c = \sqrt{3}$?

2. If $x = \frac{4ab}{a + b}$, prove that

(1) $\frac{x + 2a}{x - 2a} + \frac{x + 2b}{x - 2b} = 2$. And

(2) Find the value of $\frac{x - 2a}{x + 2b} - \frac{x + 2a}{x - 2b} - \frac{16ab}{4b^2 - x^2}$

3. Find the square root of

$$49x^4-\frac{14x^3}{5}+\frac{1051x^2}{25}-\frac{6x}{5}+9.$$

4. A coach leaves a town, travelling on the average $6\frac{1}{2}$ miles per hour; 12 hours afterwards, a horseman, who travels $8\frac{3}{4}$ miles an hour, is despatched after it. In what time will he overtake the coach?

Z.

1. Show that $1+px+qx^2+rx^3$ is a perfect cube if $p^2=3q$, $q^2=3pr$.

2. Find the G. C. M. of $x^4-px^3+\overline{q-1}x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$.

3. Simplify

$$\frac{1}{y+\cfrac{1}{1+\cfrac{y+1}{3-y}}} \text{ and } \left(x-\frac{xy}{y-\cfrac{x^2}{x-y}}\right)\left(\frac{x}{y^2}+\frac{y}{x^2}\right).$$

4. Two men start at the same time at a distance of c miles from each other, in opposite directions, and travel a and b miles a day respectively. Where and when will they meet?

Discuss your result (1) when b is negative, (2) when $b=-a$.

SECOND SERIES.

(a)

1. What is meant by the G. C. M. of two algebraical expressions? Find it for $5x^3-8x^2+x+2$ and
$5x^2-12x+7$.

2. Prove that $\frac{a+b}{ab}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{b+c}{bc}\left(\frac{1}{c}-\frac{1}{b}\right)$
$=\frac{a+c}{ac}\left(\frac{1}{a}-\frac{1}{c}\right)$.

3. Solve the equations :—

(1) $\frac{3x+1}{13}-\frac{4x-1}{5}=\frac{2-x}{2}-\frac{2x-5}{3}$.

(2) $\left.\begin{array}{l}97x-79y=320\\31y-13x=100\end{array}\right\}$.

4. Solve the equations :—

(1) $6x^2-15=x$.

(2) $2ax^2+(a-2)x=1$.

(b)

1. Divide $a^3+b^3+8c^3-6abc$ by $a+b+2c$.

2. Find the cube root of
$8b^3-84b^2x+294bx^2-343x^3$.

3. What are the three methods of solving simultaneous equations of the first degree of two unknown quantities? Solve the following by each method :—

$\left.\begin{array}{l}2x+\cdot4y=1\cdot2\\3\cdot4x-\cdot02y=\cdot01\end{array}\right\}$.

4. Show how the two values of x may be found from the equation $ax^2+2bx+c=0$, and that they will be equal if $ac=b^2$.

Solve,

$$(1)\ 3x^2-22x+35=0.$$

$$(2)\ \frac{x+7}{x+11}-\frac{x+5}{x+12}=\frac{47}{306}.$$

(c)

1. Find the cube root of $\frac{a^3}{8}-\frac{8b^3}{27}-\frac{a^2b}{2}+\frac{2ab^2}{3}$.

2. Simplify $\frac{5x-8}{7}-\frac{3x+8}{8}-\frac{\frac{19x}{2}-61}{28}$.

3. Solve the equations :—

$$(1)\ \left.\begin{aligned}\frac{2x}{3}+\frac{4y}{5}&=64\\ \frac{5x}{6}+\frac{9y}{10}&=77\end{aligned}\right\}.$$

$$(2)\ (x-3)^2+4x=44.$$

4. A vessel is filled with a mixture of spirit and water in which 70 per cent. is spirit ; 9 gallons are taken out and the vessel is filled up again with water, and now $58\frac{1}{3}$ per cent. is spirit : find how much the vessel contains.

(d)

1. Prove that $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$; and simplify

$$\left(1+\frac{x}{y}\right)\times\left(1-\frac{x}{y}\right)\div\frac{y}{x^2+y^2}.$$

2. Find the cube root of

$$343x^3-1323x^2y+1701xy^2-729y^3.$$

3. Solve the equations :—

$$(1)\ \left.\begin{aligned}\frac{3x-7y}{3}&=\frac{2x+y+1}{5}\\ 8-\frac{x-y}{5}&=6\end{aligned}\right\}.$$

$$(2)\ \frac{3x}{x+2}-\frac{x-1}{6}=x-9.$$

4. Two men working together can do a piece of work in 4 days. One works alone for 2 days, and the other then coming to assist him, it is finished in $2\frac{1}{2}$ days more. In what time can they separately do the work?

(e)

1. Find the cube root of $8x^3+36x^2y^2+54xy^4+27y^6$.
2. Solve the equations :—

$$(1)\ \left.\begin{aligned}5x-4y&=19\\ 4x+2y&=36\end{aligned}\right\}.$$

$$(2)\ 8x+\frac{7}{x}=\frac{65x}{7}.$$

3. Solve the equations :—

$$(1)\ \left.\begin{aligned}7x+3y-2z&=16\\ 2x+5y+3z&=39\\ 5x-y+5z&=31\end{aligned}\right\}.$$

$$(2)\ \frac{x}{x+3}+\frac{x+3}{x}=2\cdot 9.$$

4. If A had travelled half-a-mile an hour faster, he would have finished his journey in four-fifths of the time; if he had travelled half-a-mile an hour slower, he would have been two hours and a half longer. How many miles did he travel?

(f)

1. Show how the cube root of an algebraical expression may be found, and find the cube root of

$$x^6-3x^5y+6x^4y^2-7x^3y^3+6x^2y^4-3xy^5+y^6.$$

2. Solve the equations :—

(1) $\left.\begin{array}{l}17x-18y=31\\18x-19y=33\end{array}\right\}.$

(2) $\dfrac{x-7}{2x+1}-\dfrac{3x-2}{x-5}=\dfrac{32-11x}{2x^2-9x-5}.$

3. A vessel containing 120 gallons is filled in 10 minutes by two spouts running successively; the one runs 14 gallons in a minute, the other 9 gallons in a minute. What time did each run?

4. Solve the equations :—

(1) $5x^2-10x=75.$

(2) $27x(x-2)-21(x-1)+7=0.$

(g)

1. Find the G. C. M. of $12x^4-7x^3-6x^2+2x+1$ and $8x^4-6x^3-5x^2+2x+1.$

2. Show that the difference of the squares of the highest and lowest of any three consecutive integers is equal to four times the middle one.

3. Show how the numerical process of extracting the cube root corresponds to the algebraical, and find the cube root of 7077888 in illustration of your answer. Find also the cube root of

$$x^6-6x^5y+3x^4y^2+28x^3y^3-9x^2y^4-54xy^5-27y^6.$$

4. Solve the equations :—

$$(1)\ \left.\begin{array}{c}\dfrac{3x-7y}{3}=\dfrac{2x+y+1}{5}\\[2mm] 8-\dfrac{x-y}{5}=6\end{array}\right\}.$$

$$(2)\ \frac{48}{x+3}=\frac{165}{x+10}-5.$$

(h)

1. Find the remainder when

$x^7-10x^6+8x^5-7x^3+3x-11$ is divided by x^2-4x+3.

2. Find the square of $1+\frac{x}{2}-\frac{x^2}{8}$ and the

square root of $\frac{a^4}{64}+\frac{a^3}{8}-a+1$.

3. Solve the equations :—

$$(1)\ \frac{2x+3a}{2x-3a}+\frac{2x-3a}{2x+3a}=\frac{a+b}{a-b}+\frac{a-b}{a+b}.$$

$$(2)\ x-15\tfrac{3}{4}+\frac{5}{x-15\frac{3}{4}}=6.$$

4. Find two numbers in the proportion of 8 to 5, whose product is 360.

(i)

1. Show how the L. C. M. of three or more expressions may be found.

Find the L. C. M. of $x-2$, x^2-1, x^2-x-2.

2. Find the cube root of $x^3-\frac{1}{x^3}-3\left(x^2+\frac{1}{x^2}\right)+5$.

3. What is meant by *independent* equations? Show that the two equations which might result from the following question are not independent: 'Two men together earned 9*s.* 2*d.* in a day, but, the following day their wages were reduced 10 per cent., and their joint earnings were 8*s.* 3*d.* What were their respective wages?' Solve the following problem: 'Two men together earned 9*s.* 2*d.* in a day, but the following day the wages of one were reduced 10 per cent., and of the other 15 per cent., and they earned 8*s.* What were their respective wages?'

4. Solve the equations:—

$$(1)\quad \left.\begin{aligned} 2x-9y&=11 \\ 3x-12y&=15 \end{aligned}\right\}.$$

$$(2)\quad (x+2)^2=4x+5.$$

(j)

1. Find the cube root of $27-\frac{54}{x}+\frac{36}{x^2}-\frac{8}{x^3}$.

2. Simplify—

$$\frac{a}{(a-b)(a-c)}-\frac{b}{(b-c)(b-a)}-\frac{c}{(c-a)(c-b)}.$$

3. The denominator of a fraction is one less than the numerator, and twice the fraction added to three times its reciprocal is 5. Find it.

4. Solve the equations:—

$$(1)\quad \left.\begin{aligned} 129x+203y&=793 \\ 31x+87y&=267 \end{aligned}\right\}.$$

$$(2)\quad \frac{1}{x^2-1}-\frac{7}{8}=\frac{1}{1-x}-\frac{1}{1+x}.$$

(k)

1. Simplify—

$$\frac{3a-b}{a^2-3ab+2b^2}-\frac{a-b}{a^2-5ab+6b^2}-\frac{2a-b}{a^2-4ab+3b^2}.$$

2. Find the cube root of $\frac{27}{x^3}-\frac{54}{y}+\frac{36x^3}{y^2}-\frac{8x^6}{y^3}$.

3. Solve the equations :—

(1) $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36.$

(2) $\left.\begin{aligned} 3x-y+z &= 17 \\ 5(2x+y-z) &= 3(y+z) \\ 4(x+y+z) &= 3(1-x+3z) \end{aligned}\right\}.$

4. Solve the equations :—

(1) $x+\sqrt{5x+10}=8.$

(2) $5x-\frac{3(1-x)}{3-x}=2x-\frac{3(2-x)}{2}.$

(l)

1. Find the G. C. M. of $x^4-a(x^3-x)+(b-1)x^2-b$ and $x^4-b(x^3-x)+(a-1)x^2-a$.

2. Simplify—

$$\left(\frac{1}{a+x}+\frac{1}{a-x}-\frac{4a}{a^2-x^2}-\frac{2a}{a^2+x^2}\right)\div\frac{a}{a+x}.$$

3. Solve the equations :—

(1) $\left.\begin{aligned} 19x-21y &= 141 \\ 20x-23v &= 152 \end{aligned}\right\}.$

(2) $3x+\frac{50x}{x+1}=\frac{39(1-x)}{4}.$

4. A man rows 5 miles in three quarters of an hour with the tide, and returns against a tide half as strong in an hour and a half. Find the velocity of the strongest tide.

(m)

1. Simplify $n(n+3)-(n-p)(n-p+3)+p(p-3)$.

2. Find the L. C. M. of (x^2+y^2), (x^2-y^2), $(x+y)^2$, $(x-y)^2$, (x^3+y^3), (x^3-y^3).

3. Solve the equations :—

$$(1)\ \frac{4x-2}{x+2}-\frac{2x+3}{x-2}-\frac{x^2-16x}{x^2-4}=1.$$

$$(2)\ \frac{x}{4}-\frac{3x-2}{11x}+\frac{2(1-x)}{x}=0.$$

4. The length and breadth of a rectangular room are measured with a defective foot rule : the area appears to be 420·25 sq. ft., but is really 400 sq. ft. Find the error in the rule.

(n)

1. Find the value of $\frac{x+y-1}{x-y+1}$ when

$$x=\frac{a+1}{ab+1} \text{ and } y=\frac{ab+a}{ab+1}.$$

2. Reduce to its simplest form :—

$$\frac{x^2-9x+20}{x^2-6x}\times\frac{x^2-13x+42}{x^2-5x}\div\frac{x-7}{x^2}.$$

3. Solve the equations :—

$$(1)\ 8x+1+\frac{7}{x}=\frac{21+65x}{7}-2.$$

$$(2)\ \left.\begin{aligned} 2x-\frac{y-3}{5}&=4 \\ 3y+\frac{x-2}{3}&=9 \end{aligned}\right\}.$$

4. If 14 oz. of amalgam formed of 2 parts of gold and 3 parts of silver be worth 27 oz. of amalgam formed of 10 parts of gold and 14 parts of silver, what is the relative value of gold and silver?

(o)

1. Find the G. C. M. of x^3-x^2-2x+2 and $x^4-3x^3+2x^2+x-1$.

2. Solve the equations :—

(1) $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{x+3}{x-3}$.

(2) $x+y-z=8x+3y-6z=3z-4x-y=1$.

3. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. a head. After being at sea 20 days, she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ lb. Find the original number of the crew.

4. Two numbers, differing by 7, are together less than the sum of their squares by 32282. Find them.

(p)

1. Simplify—

$$(x^2+yz)^2+(y^2+zx)^2+(z^2+xy)^2-(yz+zx+xy)^2.$$

2. Find the cube root of—

$$64a^6-48a^5-84a^4+47a^3+42a^2-12a-8.$$

3. Solve the equations :—

(1) $\left.\begin{array}{r}3x-7y=7\\ 11x+5y=87\end{array}\right\}$.

(2) $\dfrac{14}{x-2}-\dfrac{144}{31x-135}=\dfrac{17}{x+8}$.

4. There is a number of two digits which, when doubled and thirty-six added, gives the same result as by inverting the digits, then doubling and subtracting thirty-six. Also, four times the sum of the digits is three less than the number itself. Find it.

(q)

1. Find the value of $\frac{x}{a}+\frac{x}{b-a}-\frac{a}{a+b}$ when

$$x=\frac{a^2(b-a)}{b(a+b)}.$$

2. Reduce to its lowest terms the fraction—

$$\frac{3x^3+2ax(a+x)-a^3}{3x^3+4ax(a+x)+a^3}.$$

3. Solve the equations :—

(1) $$\left.\begin{aligned}\frac{11x-9y}{8}+1-\frac{7x-3y}{10}&=5x-6y\\ 4(x-y)-\frac{8x-2y}{7}+\frac{13x-5y}{49}&=0\end{aligned}\right\}.$$

(2) $351x^2-758x+391=0$.

4. A dole of five guineas was left to be annually given to some old men and an equal number of old women; each of the former is to receive half a crown, each of the latter a shilling. How many will share in the dole?

(r)

1. What fraction is equivalent to the mixed expression $x^2+xy+y^2+\frac{x^2y+y^3}{x-y}$?

2. Solve the equations :—

(1) $(x+2)^2+7(3x-1)=6(2x+4)+(x-2)^2+3$.

(2) $$\left.\begin{aligned}32x+81y&=45\\ 28x-39y&=369\end{aligned}\right\}.$$

3. Solve the equations :—

(1) $\frac{77}{x^2}=45+\frac{64}{x}$.

(2) $\sqrt{x}-4=\frac{269-10x}{\sqrt{x}+4}$.

4. A man laid out £3 12*s.* in cloth, and having sold it at 3*s.* 6*d.* a yard, he gained as much as 4 yards cost him. How many yards did he buy?

(s)

1. Find the cube root of $\frac{1}{x^3}-3-x^3(x^3-3)$.

2. If a^3-b^3 be divisible by 3, prove that $(a+k)^3-(b+k)^3$ is also divisible by 3, k being any integer.

3. Solve the equations :—

$$(1)\quad \left.\begin{aligned}\frac{2x-25}{3}-\frac{6-y}{7}&=\frac{2(x-7)}{5}\\ \frac{29-y}{8}-\frac{3x-1}{10}&=\frac{4-y}{3}\end{aligned}\right\}.$$

(2) $\frac{2x}{x-4}+\frac{2x-5}{x-3}=8\frac{1}{3}$.

4. Solve the equations :—

(1) $\sqrt{x}+\sqrt{x-16}=8$.

(2) $x-5\sqrt{x}-14=0$.

(t)

1. Prove that

$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$.

2. Simplify $\frac{a^3-b^3}{a^4-b^4}-\frac{a-b}{a^2-b^2}-\frac{1}{2}\left(\frac{a+b}{a^2+b^2}-\frac{1}{a+b}\right)$.

3. Solve the equations :—

(1) $2x+3y-11=5\left(x-\frac{y}{3}\right)-(x-1)=0.$

(2) $\dfrac{x+2}{x-2}-\dfrac{x+1}{x-1}=\dfrac{2-\frac{4}{x}}{x-\frac{1}{x}}.$

4. Show that if a, b, be the roots of the equation

$x^2+px+q=0$, then $x^2+px+q=(x-a)(x-b)$.

If the same value of x satisfies the equations

$ax^2+bx+c=0$ and $cx^2+bx+a=0$, then

$a\pm b+c=0$.

(u)

1. If $x+a$ be the G. C. M. of x^2+px+q and

x^2+rx+s, then $a=\dfrac{q-s}{p-r}$.

2. Find the square root of

$9x^2-24xy+16y^2+12xz-16yz+4z^2.$

3. Solve the equations :—

(1) $\left.\begin{aligned} x-\frac{3x-2y}{7}&=y-\frac{4x-5y}{19} \\ y-\frac{2x-7y}{6}&=2(x-1)-\frac{3x-7}{8} \end{aligned}\right\}.$

(2) $35x^2-3x=494.$

(3) $x^2-(3a-b)x=\dfrac{6b(b^2-a^2)}{a+b}.$

4. A quadratic equation cannot have more than two roots.

Find the condition that the same value of x may *satisfy* the equations $x^2+px+q=0$ and $x^2+rx+s=0$.

(v)

1. Prove that $\dfrac{a^2+b^2+c^2+d^2}{(ab-cd)^2+(ad+bc)^2}=\dfrac{1}{a^2+c^2}+\dfrac{1}{b^2+d^2}$.

2. (1) If $a^2+c^2=2b^2$, prove that $\dfrac{1}{a+b}+\dfrac{1}{b+c}=\dfrac{2}{a+c}$.

   (2) If $a+c=2b$, and $\dfrac{2}{b}=\dfrac{1}{a}+\dfrac{1}{d}$, prove that

   $\dfrac{c}{d}=1-\dfrac{2(a-b)^2}{ab}$.

3. If α, β, represent the roots of the equation $x^2-(1+a)x+\frac{1}{2}(1+a+a^2)=0$, then $\alpha^2+\beta^2=a$.

4. A man bought a horse and carriage for £100, and sold the horse at a gain of 50 per cent., and the carriage at a loss of 25 per cent. If on the whole he gained 5 per cent., find what he paid for the horse.

(w)

1. Prove that $x^4+ax^3+a^2x^2+a^3x+a^4$ is divisible by $x^2+nax+a^2$ if $n^2=n+1$.

2. Solve the equation:—

$$\left.\begin{aligned}\frac{x}{2}+\frac{y}{4}+\frac{z}{6}&=3\\ \frac{x}{4}+y-\frac{z}{12}&=4\\ \frac{x}{5}+y+\frac{z}{10}&=5\end{aligned}\right\}.$$

3. Show that the sum of the roots of the equation $ax^2+bx+c=0$ is $-\dfrac{b}{a}$, and their product is $\dfrac{c}{a}$.

Form the equation whose roots are $\frac{3}{4}$, $\frac{4}{5}$.

4. Two persons a certain distance apart, setting out at the same time, are together in 25 minutes if they walk in the same direction, but in 15 minutes if they walk in opposite directions: compare their rates of walking.

(x)

1. Simplify $\dfrac{a+b}{b}-\dfrac{2a}{a+b}+\dfrac{a^2b-a^3}{a^2b-b^3}$.

2. If $z^2=\dfrac{ay^2-a^2}{y^2}$ and $y^2=\dfrac{ax^2-a^2}{x^2}$, find the value of

$\dfrac{az^2-a^2}{z^2}$ in terms of x.

3. Solve the equations :—

(1) $x=348-60x^2$.

(2) $4x+x\sqrt{4x+1}=\frac{55}{36}x^2-1$.

4. The sum of two numbers is $6\frac{1}{2}$, and the sum of their cubes is $128\frac{3}{8}$: find them.

(y)

1. Simplify—

$$\frac{\{(ax+by)^2+(ay-bx)^2\}\{(ax+by)^2-(ay+bx)^2\}}{x^4-y^4}.$$

2. If $2u=x+\dfrac{1}{x}$, $2v=y+\dfrac{1}{y}$, prove that

$$uv+\sqrt{(u^2-1)(v^2-1)}=\tfrac{1}{2}\left(xy+\frac{1}{xy}\right).$$

3. Solve the equation :—

$$x^2-x+5\sqrt{2x^2-5x+6}=\frac{3x+33}{2}.$$

4. A, B, C do a piece of work together in a certain time. A could have done it alone in 6 hours; B, in 15 hours more; C, in twice the time. How long did it *occupy them?*

(z)

1. Find the G. C. M. of $3x^2-(4a+2b)x+4ab-b^2$ and $x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$.

2. If $s=\frac{1}{2}(a+b+c)$, prove that

$$\frac{1-\dfrac{b^2+c^2-a^2}{2bc}}{1+\dfrac{b^2+c^2-a^2}{2bc}}=\frac{(s-b)(s-c)}{s(s-a)}.$$

3. Solve the equations :—

$$(1)\ \frac{3}{1+\sqrt{x}}+\frac{3}{1-\sqrt{x}}=4.$$

$$(2)\ x-5\sqrt{x}-14=0.$$

4. Prove that the roots of $x^2+px+q=0$ will be real and equal if $p^2=4q$, real and unequal if p^2 be greater than $4q$, imaginary if p^2 be less than $4q$.

If t be the ratio of the roots, show that

$$\left(\sqrt{t}+\sqrt{\frac{1}{t}}\right)^2=\frac{p^2}{q}.$$

ANSWERS.

(1) 4. 131. 5. 33. 6. -21. 7. 284.
8. -140. 9. 2688. 10. -6820.

(2) 1. $1\frac{1}{3}$. 2. $-1\frac{19}{36}$. 3. $44\frac{283}{2275}$. 4. 19. 5. 112.
6. 222. 7. 138. 8. 120. 9. -12.
10. 0.

(3) 1. $3b$. 2. a. 3. 6. 4. $a^2+b^2+c^2$.
5. $\frac{31ab^2}{7}-\frac{33a^2b}{5}$. 6. $\frac{85x^3}{9}+\frac{5x^2y}{2}-\frac{8y^3}{17}$.

(4) 1. $2a-2c$. 2. $a^2-34ab-c^2$. 3. $\frac{x^2}{2}+\frac{3y^2}{5}$.

(5) 1. $(p+q)x+(p+q)y$. 2. 0. 3. $3ax^2-5bx$.
4. $(p^2-2q^2)x^2-p^2x-2r^3$.

(6) 1. $-4b$. 2. y^2+z^2. 3. $-26a+19b+5c+d$.
4. $-x-y$.

(7) 1. $15a^9b^4c^7d$; $-42a^5b^4$; $\frac{4}{5}m^5a^9b$. 2. $45a^2-8ab-21b^2$.
3. $9a^4-21a^3x+15a^2x^2-35ax^3$. 4. $\frac{a^2}{7}-\frac{283ab}{630}+\frac{b^2}{3}$.
5. $12a^3-17a^2x+26ax^2-15x^3$.
6. $21x^3-40x^2y+7xy^2+12y^3$.
7. $16x^4-16x^3y+12x^2y^2-4xy^3-3y^4$.
8. $30a^2+69ab+27b^2$; $6a^2-41ab+63b^2$.
9. $a^2-b^2+2bc-c^2$; $b^2-a^2-2ac-c^2$.
10. $59x^2-13ax-52a^2$.
11. $a^3+3a^2x+3ax^2+x^3$; $8x^3-36ax^2+54a^2x-27a^3$.
12. $b-a$.
13. $81m^4x^4-216m^3nx^3y+216m^2n^2x^2y^2-96mn^3xy^3$
$+16n^4y^4$. 14. $625a^4$

17. $(3x+2a)(3x-2a)$; $(x+1)(x+2)$; $(2x+1)(x+3)$; $(2x+3)(3x+4)$.

18. $(x+y)(x-y)(x^2+y^2)$; $(x-1)(x-2)$; $(2x+1)(x-3)$; $(2x-3)(3x+4)$.

19. $(4x+5y)(4x+5y)$; $(3x-5y)(7x-2y)$; $8(x-y)(2x-5y)$.

(8) 1. $5bc$; $8ac^2$; $-5a^2b$. 2. $3ax-15$; $-5a^5x^2+7a^3$.
3. $a+b$; $a-b$; $a-b$; a^2-ab+b^2 ; $a-b$.

(9) 1. $4x-7$; $3x-9$. 2. $6a+7b$; $5a-4b$.
3. $12x^2+43x+35$; $8x^2-10x-25$.
4. $2x^2y^2+2xy+1$. 5. $32+16x+8x^2+4x^3+2x^4+x^5$.
6. x^2-ax+b^2. 7. $a^2+b^2+c^2-ab-ac-bc$.
8. $1+3x+9x^2+27x^3$. 9. $3x^2-\frac{2xy}{3}+\frac{3}{4}y^2$.

(10) 1. $12bc^3$. 2. $3x$. 3. ab. 4. $a-b$. 5. $x+y$.
6. $x-4$. 7. $2x-3$.

(11) 1. $x-2$. 2. $x-5a$. 3. x^2-ax+a^2. 4. x^5-3.
5. $x-2$. 6. $x-3$. 7. $x-7$. 8. a^2-3.
9. $2x^3-4x^2+x-1$. 10. a^2-9b^2. 11. $a(2a-3b)$.
12. $x-p$. [illegible] 14. $30a^2b^6c^3d$.
15. $42xy($[illegible]
16. $48xy(x$[illegible]
17
18
19

Erratum

2C

P. 117, (15) 19, *for* $-\frac{2}{3}$ *read* $\frac{-41}{6}$.

(12) 1

4. $\frac{x(x-y)}{3y^2(x^2-xy+y^2)}$; $\frac{a-3b}{a+2b}$. 5. $\frac{3(a^2-b^2)}{4ab}$; $\frac{-2b^3}{a^2-b^2}$.

6. $\frac{x^2+a^2}{(x^2-a^2)(x-a)}$. 7. $\frac{-2}{x(4x^2-1)}$; $\frac{1}{x^2(x^2-1)}$.

8. $\frac{4(x+a)}{6x^2-13ax+6a^2}$. 9. $\frac{25ax}{28by}$; $\frac{35a^2}{36bcx}$. 10. $\frac{x(a+x)}{a(a-x}$

11. $\frac{2b-17a}{36}$. 12. 0. 13. $\frac{7x-5}{x^2-x+1}$.

14. 1. 15. $\dfrac{x}{(x-2a)^2}$. 16. 1.

17. 0. 18. $\dfrac{x-b}{x+b}$. 19. $\dfrac{x-y}{x+y}$.

20. $\dfrac{x^3-3x+2}{x^2}$. 21. $x+1$.

(13) 1. $2xy^2z^4$; $9a^3b^6$; $\dfrac{3a^2b^3}{4cd}$; $\dfrac{5a^4b^5}{6x^3y^7}$.

2. $5a-4b$; $7c-8d$; $9x-11y$.

3. x^2-xy+y^2. 4. x^2-3x+2.

5. $5a^2-4ab+9b^2$. 6. $\dfrac{x}{y}-\dfrac{y}{x}$.

7. $6x^2-3x+\frac{2}{3}$. 8. $a+2b+3c$.

9. $4a^3-a-2$. 10. $\dfrac{a^2+b^2+c^2}{abc}$.

(14) 1. $a-2b$. 2. $3x+4y$. 3. $5a-7b$.

4. $9x+11y$. 5. $2a^2x^2-7by^3$.

6. x^2-2x+1. 7. $x-1$. 8. 29; 99.

9. 12; 13. 10. $4a^2-a-2$.

(15) 1. 10; 1. 2. -7; 5. 3. $\frac{1}{6}$; $\frac{1}{7}$. 4. 4; $\dfrac{6a}{7}$.

5. $\dfrac{-ab}{a+b}$; $\dfrac{bc}{ad}$. 6. $2m+3n$; $\dfrac{m+n}{2}$;

7. 7; 7. 8. -1; $5\frac{1}{2}$. 9. 10; $3\frac{3}{13}$. 10. 11.

11. 5. 12. 4. 13. 5. 14. 8.

15. ·7. 16. 4. 17. -107. 18. 72.

19. $-\frac{2}{3}$. 20. $\dfrac{a^2c}{b^2}$. 21. $3\frac{3}{8}$. 22. 37.

23. 4. 24. $1\frac{3}{14}$.

(16) 1. $1\frac{1}{2}$, $10\frac{1}{2}$. 2. 42.

3. 100 shillings and 20 half-crowns. 4. 40.

5. £9,000. 6. £270. 7. 24lbs. 8. 20 gallons.

9. $5\frac{1}{7}$ miles per hour; $4\frac{13}{14}$ miles per hour.

10. $\dfrac{2abmn}{am-bn}$. 11. 12 and 4. 12. 525 and 385.

13. £33 each. 14. $4\frac{1}{2}$ miles per hour.

15. 6 and 12 miles per hour. 17. $5'\,27\frac{3}{11}''$ past one.

18. $27'\,16\frac{4}{11}''$ past two. 19. $10\frac{10}{11}'$ to four.

20. $3\frac{1}{3}$ days. 21. $19\frac{17}{37}$ minutes.

22. $1\frac{1}{9}$ days afterwards. 23. 25. 24. 54.
25. 560. 26. $1281\frac{21}{59}$ days ; 75600 days. 27. 54.
28. 98lbs. 29. 505. 30. 36 minutes.

(17) 1. 8, 2. 2. 7, 4. 3. 4, -1. 4. 6, 0.
5. 8, 5. 6. 31, 32. 7. ·02, 2·9. 8. 8, 5.
9. 5, 3. 10. 12, 6. 11. 6, 12. 12. 6, 7.
13. b, a. 14. $\dfrac{ac(dn-bm)}{ad-bc}$, $\dfrac{bd(am-cn)}{ad-bc}$.
15. $a-2b$, $2a-b$. 16. $\frac{1}{2}$, $\frac{1}{3}$. 17. $\frac{25}{23}$, $\frac{15}{17}$.
18. $\frac{1}{2}$, $\frac{1}{3}$. 19. -11, 8.
20. $\dfrac{a^2}{a-b}$, $\dfrac{-b^2}{a-b}$. 21. $\dfrac{c(a^2+b^2)}{a^2-b^2}$, $\dfrac{c(a^2+b^2)}{2ab}$.
22. $\dfrac{a^2+bc}{ac}$, $\dfrac{a^2+bc}{c^2}$.

(18) 1. 4, -5, 6. 2. 16, 12, 10. 3. 7, 8, 9.
4. 3, 5, 4. 5. 12, 12, 12.
6. $\frac{2}{3}(a+b+c)-a$, &c. 7. 1, -2, 3.
8. 7, 20, 28.

(19) 1. 94 half-crowns, 29 shillings.
2. 20 persons, 9 shillings. 3. 5*s*. 6*d*. and 4*s*.
4. $\frac{7}{19}$. 5. 15, 25.
6. 20 men, 25 women. 7. 6 L. R., 8 S. R.
8. 200 at 3*s*., 100 at 4*s*. 9. 35. 10. 48.
11. $112\frac{1}{2}$ and 90. 12. 860.
13. 100 florins, 60 half-crowns. 14. 150 yards.
15. 100 miles. 16. 600, 700 per mille.
17. 1080 yards, $16\frac{1}{2}$ minutes. 18. 567.
19. 30 and 50 miles per hour. 20. 13 and 15 miles.

(20) 1. ± 13. 2. ± 17. 3. $\pm\frac{1}{5}$. 4. 4, $-3\frac{4}{7}$.
5. $-2\frac{1}{2}$, $-7\frac{1}{2}$. 6. 1, $-\frac{3}{11}$. 7. $\frac{3}{4}$, $-\frac{1}{12}$; 8, $-5\frac{1}{3}$.
8. -3, -22. 9. $1\frac{1}{2}$, $1\frac{1}{3}$; 2, $-\frac{2}{3}$. 10. $\frac{1}{5}$, $\frac{1}{6}$; 2, -3.
11. $9\frac{15}{17}$, -11. 12. 9, $\frac{1}{14}$. 13. 8, $-6\frac{2}{3}$.
14. $\dfrac{1}{a}$, $-\frac{1}{2}$. 15. m, $-n$. 16. 4, -1.
17. $-a$, $-b$. 18. $\dfrac{a^2}{b}$, $\dfrac{-2a^2}{b}$. 19. $\dfrac{a}{b}$, $\dfrac{b}{a}$.
20. $\dfrac{a+b}{c}$, -1.

(21) *1.* $x^2-\frac{9}{2}x-9=0$. 2. $x^2-7x=0$. 3. 4. 4. $-2b$.
5. *3*, $-\frac{4}{5}$. 6. 4, $-2\frac{57}{67}$. 7. 6, -15.

8. $\pm\sqrt{\dfrac{b^2-ac}{a-b}}$. 9. -4, -5.

10. 3, $5\frac{13}{14}$. 11. $a-3b$, $6a-3b$. 12. ± 4, ± 3.
13. $\pm 1\frac{2}{3}$, $\pm\frac{2}{3}$. 14. -2, $\pm\sqrt{7}+2$. 15. 1, 2, 3.
16. 4, -9. 17. 11. 18. $18\frac{1}{2}$, 5.
19. $1\frac{1}{3}$. 20. 3. 21. 7.
22. ± 2, ± 3. 23. a, $-2a$ 24. $\dfrac{a}{5}$.

25. $2a^2\sqrt{\dfrac{3}{a^2+1}}$. 26. $\frac{4}{81}$.

27. $\dfrac{b^2}{9a}$. 28. $\dfrac{-p}{q}$.

(22) 1. 25, 26. 2. $x(x+1)+\frac{1}{4}=(x+\frac{1}{2})^2$. 3. 8, 9.
4. 1, 9. 5. 75. 6. 60. 7. 8, 12.
8. 4 miles an hour. 9. 288, 224.
10. 30, 25. 11. 3 and 5 hours. 12. 104.
13. 8, 16. 14. 8 pence. 15. 16. 16. 16.
17. $1\frac{1}{2}$ pence. 18. 9 gallons. 19. 24 days.
20. 60 acres. 21. 18. 22. 6 and 9 days.
23. 40. 24. £70. 25. 150.
26. 4 miles an hour. They meet 54 miles from A's starting point. 27. 12 or 8.

EXAMINATION PAPERS.

First Series.

A. 1. $4ab-a^2-5b^2$; $a+10b-13c+5d$. 2. $a^2-(3a+b)$.
3. $\dfrac{a-b}{(a+b)^2}$; $\dfrac{2}{x-3y}$. 4. 5 ; $\dfrac{a+b+c}{ab+ac+bc}$.

B. 1. $37(a+b)$, $\pm 71(a-b)$.
2. $x^4+2x^3-7x^2-8x+12$; x^2-3x+2.
3. 0. 4. 4 ; $-\frac{1}{8}$.

C. 1. 5. 2. $3a^2-2axy-2x^2y^2$. 3. $a-3b$. 4. 3 ; 9.

D. 1. -921 ; $-\frac{19}{216}$. 2. x^2+2x-3.
3. $\dfrac{x-1}{x+2}$; $\dfrac{x^2}{x-2}$. 4. $\frac{545}{4986}$; $\frac{4423}{4986}$.

E. 1. $90a$. 2. a^5+32b^5. 3. $2x-3a$. 4. $1\frac{1}{4}$; 5.

F. 2. $12x^4-31ax^3+48a^2x^2-71a^3x+45a^4$; a^2-1.

3. $\frac{3a^2+2b^2}{5a(2a+3b)}$. 4. $\frac{a+2c}{b-1}$; 8.

G. 1. $\frac{1}{2a}+\frac{x}{4a^2}+\frac{x^2}{8a^3}+\frac{x^3}{16a^4}+$ &c;

$qx^3-(1+p+pq)x^2+(p+p^2+q^2)x-(q+pq)$.

2. $\frac{3y}{x(x+y)}$. 3. $x^2-x+\frac{1}{4}$. 4. 13; 8.

H. 2. x^4-1. 3. $\frac{4a^3-3x^3}{(a-x)(a^3+x^3)}$. 4. 16; 51.

I. 1. $2a+3b-4c$; $x^2+2xy-5xz+4y^2+25z^2+10yz$.

2. $\frac{2a^3}{x^4+a^2x^2+a^4}$. 3. $\frac{2}{3}$; 9. 4. 48.

J. 1. $8y-12x$; $64y^2-144x^2$. 2. x^2+7; $5x^2-1$.
3. $6x^2-11xy+13y^2$. 4. 8; $-\frac{7}{10}$.

K. 1. 0; $-2b(x+y)$. 2. $\frac{5(4a^2+1)}{6(5a^2+a+1)}$.

3. $x+1-\frac{1}{x}$. 4. $\frac{2}{3}$; $\frac{b^3-a^3}{b^2+a^2}$.

L. 1. $2bc+2ab-2b^2$; 0. 2. x^2-x-1. 3. 0. 4. 7; $3\frac{1}{2}$.

M. 1. x^2-2x+3; $1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}, \frac{b^4}{a^3}$. 2. x^2-3x+2.

3. $\frac{8xy(x^2+y^2)}{(x^2-y^2)^2}$; $\frac{2(x^2+y^2)}{x^2-y^2}$; $\frac{4xy}{x^2-y^2}$. 4. 3.

N. 1. x^2+7x+9; $1+2x+3x^2+4x^3+5x^4$.
2. $x-4$; x^2-3x-4; $x-4$.
3. 20; $2\frac{1}{2}$. 4. Eldest 6*s.* 6*d.*, &c.

O. 1. $(6x-7y)(6x+7y)$; $(4x-5y)^2$; $(x-5y)(x-7y)$;
$(2x-y)(x^2-3y^2)$.

2. $9a^3b^2(a-1)$. 3. $\frac{1-b}{a}$. 4. $33\frac{4}{11}$; $\frac{3}{31}$.

P. 1. $y^{12}+b^4y^8-b^8y^4-b^{12}$; $(y^4+b^4)^2$. 2. xy; x^3-7x+6.

3. x^2-2x+3; $\frac{x^2}{a^2}+\frac{x}{a}-\frac{a}{x}+\frac{a^2}{x^2}$. 4. 7; 2.

Q. 2. $x+6$. 3. 5; 72. 4. £4,000 stock.

R. 1. $2-5x$; 127. 2. x^2-y^2. 3. $\frac{x}{(x-2a)^4}$. 4. 15, 5.

S. 1. $105xy-25x^2-5y^2$; $4a^4-5a^2b^2+b^4$. 2. $\frac{a+x}{x-y}$.

3. 17; 4. 4. $\frac{ap+bq+cr}{a+b+c}$.

T. 1. $a^4-a^2b^2+b^4$; $a^8-2a^6b^2+3a^4b^4-2a^2b^6+b^8$.

2. x^2-3x+1. 3. $\frac{c+d}{f+2x}$; $\frac{2y^2+3y-5}{7y-5}$.

4. 320 and 440 yards per minute.

U. 1. $\frac{x^4}{4}+\frac{8x^2y^2}{9}+y^4$; $1+2ab+2a^2b^2$.

2. $(x-2)(x^2-3)(x^2-4)$. 3. $\frac{x(1-3x)}{(1-x)^3}$; $\frac{1}{b(a-b)}$.

4. $\frac{5n-4p}{2}$, $\frac{4p-3n}{2}$.

V. 2. $b+c-a$. 3. $3\frac{1}{4}$; $-\frac{4}{5}$. 4. 100.

W. 1. $(x-a)(x-b)$. 2. $12\frac{1}{24}$. 3. 1.
4. 5875 × 96000 miles per second.

X. 1. px^2+qx-r. 3. 2; 7. 4. 100 at $5\frac{1}{2}d.$, 110 at $5d$.

Y. 1. $\frac{1}{a^2}+\frac{2}{ac}+\frac{1}{c^2}-\frac{1}{b^2}$; 1. 2. 0.

3. $7x^2-\frac{x}{5}+3$. 4. 36 hours.

Z. 2. (x^2-1). 3. $\frac{4}{3(y+1)}$; $\frac{x(x+y)}{y^2}$.

4. They meet in $\frac{ac}{a+b}$ days.

Second Series.

a. 1. $x-1$. 3. 4; 9, 7. 4. $1\frac{2}{3}$, $-1\frac{1}{2}$; $\frac{1}{a}$, $-\frac{1}{2}$.

b. 1. $a^2-ab+b^2-2ac-2bc+4c^2$. 2. $2b-7x$.
3. ·02, 2·9. 4. $2\frac{1}{3}$, 5; 6, $\frac{22}{47}$.

c. 1. $\frac{a}{2}-\frac{2}{3}$. 2. $\frac{1}{28}$. 3. 60, 30; 7, −5. 4. 54.

d. 1. $\frac{y^4-x^4}{y^3}$. 2. $7x-9y$.

3. 13, 3; 10, $-1\frac{4}{7}$. 4. $5\frac{1}{3}$, 16.

e. 1. $2x+3y^2$. 2. 7, 4; $\pm 2\frac{1}{3}$.
3. 2, 4, 5; 2, -5. 4. 15.

f. 1. x^2-xy+y^2. 2. 5, 3; ± 1.
3. 6 and 4 minutes. 4. 5, -3; $2\frac{1}{3}$, $\frac{4}{9}$.

g. 1. $4x^2-x-1$. 3. 192; $x^2-2xy-3y^2$. 4. 13, 3; 5, $5\frac{2}{5}$.

h. 1. $-1667x+1651$. 2. $1+x-\frac{x^3}{8}+\frac{x^4}{64}$; $\frac{a^2}{8}+\frac{a}{2}-1$.

3. $\pm\frac{3a^2}{2b}$; $20\frac{3}{4}$, $16\frac{3}{4}$. 4. 24, 15.

i. 1. $(x-2)(x^2-1)$. 2. $x-1-\frac{1}{x}$.

3. 4*s*. 2*d*., 5*s*. 4. 1, -1; ± 1.

j. 1. $3-\frac{2}{x}$. 2. $\frac{2a}{(a-b)(a-c)}$. 3. $\frac{3}{2}$. 4. 3, 2; 3, $-\frac{5}{7}$.

k. 1. $\frac{3ab}{(b-a)(2b-a)(3b-a)}$. 2. $\frac{3}{x}-\frac{2x^2}{y}$.

3. 9; 4, 0, 5. 4. 3; 4, -1.

l. 1. x^2-1. 2. $\frac{4a^2}{x^3-ax^2+a^2x-a^3}$.

3. 3, -4; $\frac{3}{17}$, $-4\frac{1}{3}$. 4. $2\frac{2}{9}$ miles an hour.

m. 1. $2np$. 2. $(x^4-y^4)(x^6-y^6)$. 3. 2; 8, $1\frac{1}{11}$. 4. $\frac{1}{41}$ ft.

n. 1. a. 2. $x-4$. 3. $\pm 2\frac{1}{3}$; 2, 3. 4. 14 to 1.

o. 1. $x-1$. 2. 0, $1\frac{2}{3}$; 2, 3, 4. 3. 40. 4. 131, 124.

p. 1. $x^4+y^4+z^4$. 2. $4a^2-a-2$. 3. 7, 2; 9, $8\frac{38}{237}$. 4. 59.

q. 1. 0. 2. $\frac{3x-a}{3x+a}$. 3. 11, 9; $1\frac{4}{13}$, $\frac{23}{27}$. 4. 30 of each.

r. 1. $\frac{x^2(x+y)}{x-y}$. 2. 2; 9, -3. 3. $\frac{7}{9}$, $-2\frac{1}{5}$; $25\frac{10}{11}$. 4. 24.

s. 1. $\frac{1}{x}-x^2$. 3. 17, 13; 6, $3\frac{1}{13}$. 4. 25; 49.

t. 2. 0. 3. 1, 3; $\frac{4}{5}$.

u. 2. $3x-4y+2z$. 3. $5, 4$; $3\frac{4}{5}, -3\frac{3}{7}$; $3(a-b), 2b$.
4. $(q-s)^2=(r-p)(rq-ps)$.

v. 4. £40.

w. 2. 2, 4, 6. 3. $20x^2-31x+12=0$. 4. 4 to 1.

x. 1. $\dfrac{b}{a+b}$. 2. x^2. 3. $2\frac{2}{5}, -2\frac{5}{12}$; 6. 4. $1\frac{1}{2}$ and 5.

y. 1. a^4-b^4. 3. $3, -\frac{1}{2}$. 4. 3 hours.

z. 1. $x-b$. 3. $-\frac{1}{2}$; 49.

LONDON : PRINTED BY
SPOTTISWOODE AND CO., NEW-STREET SQUARE
AND PARLIAMENT STREET